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Jacob Stewart

Escape and Pursuit of an L^1 Pursuer and an L^2 Target

Faculty Sponsor
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Abstract

In mathematical pursuit and escape games, a pursuer (agent 1) tries to catch a target (agent 2) by closing the distance between them. Both pursuer and target can move as long as their movement is unobstructed. The effects of limiting one of these agents to movement measured in the L^1 metric and the addition of a finite, straight-line barrier were investigated. It was found that even with lower speed (up to a ratio of $\frac{1}{\sqrt{2}}$), an L^2 target can escape from an L^1 pursuer if the target takes the "optimal" path. When introducing the barrier, there are three cases for the L^1 pursuer's path: one axis of motion (AoM) blocked, both AoM blocked, and no AoM blocked. The cases with single AoM and both AoM being blocked results in global change of the dominance regions, whereas the case with no AoM blocked only results in local change.

Introduction

Pursuit and Escape (PE) games are used to define the behavior of two or more agents in which one or more pursuer is attempting to capture one or more target (Nahin 1981). These games are usually done in a two-dimensional case to simplify modeling. There are a variety of ways to describe the way in which two agents participating in a game interact with one another, but this paper describes interactions between agents whose movement are described with respect to two different norms.

DEFINITION 1 (L^1 and L^2 Norm). *Let $P = (x, y)$ be a point in $\mathbb{R}^2$. Then,*

$$(1)\ ||P||_1 = ||(x,y)||_1 = |x| + |y| \quad \textit{is the } L^1 \textit{ norm of } (x,y)$$

$$(2)\ ||P||_2 = ||(x,y)||_2 = \sqrt{x^2+y^2} \textit{ is the } L^2 \textit{ norm of } (x,y)$$

DEFINITION 2 (L^1 and L^2 Metric). *Let $P = (x_1, y_1)$, $Q = (x_2, y_2)$ be points in $\mathbb{R}^2$. Then,*

(1) *The L^1 metric of P and Q is given by* $d_1(P,Q) = ||P-Q||_1 = ||(x_1-x_2, y_1-y_2)||_1$

(2) *The L^2 metric of P and Q is given by* $d_2(P,Q) = ||P-Q||_2 = ||(x_1-x_2, y_1-y_2)||_2$

A metric is also called a distance because it measures how far apart two points are. The L^2 metric is commonly referred to as the Euclidean metric or distance.

DEFINITION 3 (Apollonian Circle). *For two distinct points A and B in $\mathbb{R}^2$, the set of all points $P = (x, y)$ such that, for some constant $c > 0$,*

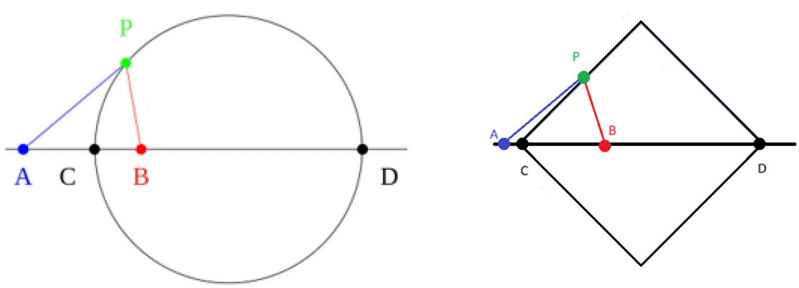
$$d_{1,2}(P,A) = c d_{1,2}(P,B)$$

is called an Apollonian Circle.

REMARK. *Apollonian Circles are circles in the classical sense, as illustrated in Figure 1.1a.*

REMARK. *Changing to the L^1 metric, i.e $d = d_1$, results in diamond shaped Apollonian 'circles.' As illustrated in Figure 1.1b.*

This paper has two objectives: 1) analyzing the movement of two agents, one being limited to movement measured with respect to the L^1 metric and the other with respect to the L^2 metric and 2) illustrating the effect of a finite, straight line barrier on the agents' movement and interactions.



(A) Apollonian circle definition for the L^2 metric in which P represents the point that has a ratio constant for $c = \frac{d_2(P,A)}{d_2(P,B)}$.

(B) Apollonian circle definition for the L^2 metric in which P represents the point that has a ratio constant for $c = \frac{d_1(P,A)}{d_1(P,B)}$.

Figure 1.1 Apollonian Circle in L^2 and L^1 Metric

These objectives are motivated by certain predator-prey interactions, as well as military applications like unmanned ground vehicle (UGV) to unmanned aerial vehicle (UAV) pursuit and target-tracking algorithms. The most interesting cases arise with an agent on the ground in a city with a grid-like structure, representing the L^1 metric case, and an aerial agent that is free to move above the city, representing the L^2 metric case.

The literature covers a substantial amount of work on traditional escape and pursuit games, particularly involving single target, single pursuer games, shown in (Nahin 1981). This is not necessarily new, but a paper by (Oyler et al. 2016) expands upon previous works to add complexity to the problem of barrier addition and how dominance regions behave around said barrier among other contributions that are less relevant to this paper.

NOTE. For the duration of this paper, we consider two agents: Agent A (the pursuer) and Agent B (the target). Agent A is restricted to movement measured with respect to the L^1 metric ($d_A = d_1$) with starting point (x_A, y_A) and constant speed $v_A > 0$. Agent B follows for the L^2 metric ($d_B = d_2$) with starting point (x_B, y_B) and constant speed $v_B > 0$.

DEFINITION 4 (Isochronic Curve). *Let $\mathcal{U}$ be a region in $\mathbb{R}^2$ and $(x_o, y_o) \in \mathcal{U}$. An Isochronic Curve in $\mathcal{U}$ is a continuous curve $(x(t), y(t)) : [0, T] \to \mathbb{R}^2$, such that*

$$\begin{cases} (x(0), y(0)) = (x_o, y_o) & \\ d((x(t), y(t)), (x_o, y_o)) \leq vt & \text{for} \quad 0 \leq t \leq T \\ (x(t), y(t)) \in \mathcal{U} & \text{for} \quad 0 \leq t \leq T \end{cases} \tag{1.1}$$

where d is either the L^1 or L^2 metric. If equality holds above in (1.1)*, the corresponding isochronic curve is called an isochrone.*

DEFINITION 5 (Isochronic Set). *Let $\mathcal{U}$ be a region in $\mathbb{R}^2$, $(x_o, y_o) \in \mathcal{U}$, and $v > 0$. For $T > 0$, the set*

$$I(T) = \{(x, y) \in \mathcal{U} \mid \exists \text{ an isochronic curve in } \mathcal{U}, (x(t), y(t)), 0 \leq t \leq T \text{ such that}$$
$$(x(0), y(0)) = (x_o, y_o), (x(T), y(T)) = (x, y)\}$$

The set $I(T)$ is called the Isochronic Set at time T.

For Agent A, we consider $I(T)$ with $d = d_A$, $v = v_A$, $(x_o, y_o) = (x_A, y_A)$ and write $I_A(T)$. For Agent B, we consider $I(T)$ with $d = d_B$, $v = v_B$, $(x_o, y_o) = (x_B, y_B)$ and write $I_B(T)$.

REMARK. *If $\mathcal{U} = \mathbb{R}^2$ the set $I(T)$ reduces to*

$$I(T) = \{(x, y) \in \mathcal{U} \mid d((x, y), (x_o, y_o) \leq vt, 0 \leq t \leq T\}$$

This will define circles if $d = d_2$ and diamond shapes if $d = d_1$.

Dominance Regions are areas over which an agent is able to reach any point in that are before another agent.

DEFINITION 6 (Dominance Region). *The Dominance Region of Agent B is defined as*

$$\bigcup_{t^* > 0} I_B(t^*) \backslash I_A(t^*)$$

REMARK. *We interpret an unbounded Dominance Region for Agent B to mean that Agent B can always reach points at any time $t^* > 0$ which cannot be reached by Agent A for the same time, t^*.*

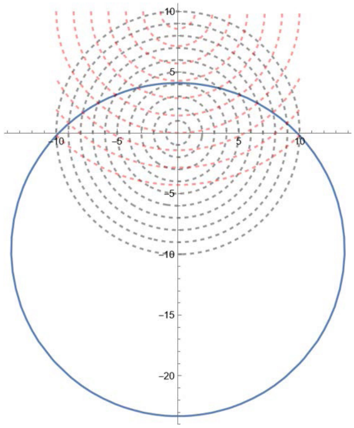


Figure 1.2
Agent one (located at the origin) has a bound dominance region because the intersection of the two agent's isochrones (dotted lines) defines a closed circle.

EXAMPLE 1. **Figure 1.2** *shows two agents whose movement are described with respect to the L^2 metric on a two dimensional plane, one positioned at the origin, and the other ten units away along the y-axis. The agent starting at the origin is given a lower starting speed than the other. This results in its dominance region being bounded because the agent is unable to escape the other. This is because they both have their movement measured with respect to the same metric.*

NOTE. *This paper explores the changes when one agent's movement is measured with respect to the L^2 metric and the other's is measured with respect to the L^1 metric. This gives the agent moving with respect to the L^2 metric an unbounded dominance region even if it is slower than the other agent, up to a later defined velocity ratio.*

This leads to the main result of this work:

THEOREM (L^1-L^2 Norm Interaction Theorem). *Suppose Agent A has starting point (c, d), for $c \geq 0$ and $d > 0$, and Agent B has starting point $(0, 0)$. For a velocity ratio, $\gamma = \frac{v_A}{v_B} \geq \frac{1}{\sqrt{2}}$, then the Dominance Region of Agent B, defined by $\bigcup_{t^*>0} I_B(t^*)\backslash I_A(t^*)$, is unbounded.*

The proof of this theorem is given in the following section. The last section of this work will consider the impact of a single, finite straight-line barrier on the Dominance Regions of both agents through numerical investigation. A rigorous statement regarding this is left for future work.

Theorem Background and Proof

This project started through modeling the movement of two agents using isochrones to define their dominance regions. After limiting one agent to movement measured with respect to the L^1 metric, the equations for these isochrones were defined and used to describe new dominance regions. Then, the asymptotic behavior of the curves defining the dominance region was identified, proving that the target can escape for certain 'velocities' lower than the pursuer's velocity.

To start, some results from Oyler's paper were recreated with both agents having their movement measured with respect to the L^2, or Euclidean, norm (Oyler et al. 2016). Desmos was used to plot the initial graphics for both agents. Later, Wolfram Mathematica was used to plot the more complex cases. Note that all cases were computed with $\gamma = \frac{v_A}{v_B} = \sqrt{0.6}$ and a separation of $d = 5$ for purposes of illustration. γ is the ratio of velocities between the two agents, and d is the separation between them along the y-axis.

Movement Measured with respect to the L^2 Metric

With both agent's movement being measured with respect to the L^2 metric, the distance traveled by an agent is defined using Cartesian coordinates as $x^2 + y^2 = v^2t^2$, with x and y being the coordinates of the agent and v being the velocity of the agent. The previously mentioned isochrone corresponds to a circle for every value of t, meaning that it would take the agent the same amount of time, t, to reach any point on that circle, provided it starts from the center of the circle. To model the dominance region, the intersection of the agent's isochrones are found. To do this, the left hand sides of both agent's movement equations are set equal to one another.

Both agents are each given different starting points and speeds, then the time component is removed because their movement can be defined entirely with these movement functions and the resulting isochrones. First, we define the isochrone equations of Agent A and Agent B: Agent A, $x_A^2 + y_A^2 = v_A^2t^2$ and Agent B, $x_B^2 + y_B^2 = v_B^2t^2$. Agent A's starting location, (x_A, y_A), should be different from Agent B's, (x_B, y_B). We start Agent B at the origin and Agent B at (c, d) with $c \geq 0$ and $d > 0$. We let $\gamma = \frac{v_B}{v_A}$, $x = x_B = x_A$, and $y = y_B = y_A$, then this simplifies to

$$x^2 + y^2 = \gamma((x-c)^2 + (y-d)^2) \tag{2.1}$$

Figure 1.2 shows how the isochrones of each agent intersect to from a circle. The solid line circle is enclosed, meaning that Agent B is not able to escape. This defines a bounded dominance region.

Limitation to the L^1 Norm

One agent is limited to movement measured in the L^1 norm to study its interactions with another agent that has its movement measured with respect to the L^2 norm. Limiting one agent to movement measured with respect to the L^1 norm results in the corresponding equation denoting maximal distance traveled in time t becoming $|x-c| + |y-d| = vt$. In this case, the isochrones will change their shape from being circles to being diamond shaped.

We concentrate on the case where $\mathcal{U} = \mathbb{R}^2$. $I_B(t^*)$ has as its boundary all points (x, y) such that $x^2 + y^2 = v_B^2 t^{*2}$, and $I_A(t^*)$ has as its boundary all points (x, y) such that $(|x-c| + |y-d|)^2 = v_A^2 t^{*2}$.

Hence, to understand the Dominance Region, we focus on the point (x, y) that are shared by $I_A(t^*)$ and $I_B(t^*)$, i.e. (x, y) such that

$$x^2 + y^2 = \gamma^2((x-c)^2 + 2|x+c||y-d| + (y-d)^2) \tag{2.2}$$

This equation plots the shared boundary of $I_A(t)$ and $I_B(t)$ for any $t > 0$, and, depending on the value of gamma, will lead to a bounded or unbounded dominance region for Agent B.

Proof for Theorem

L^1-L^2 Norm Interaction Theorem

Suppose Agent A has starting point (c, d), for $c \geq 0$ and $d > 0$, and Agent B has starting point $(0, 0)$. For a velocity ratio, $\gamma = \frac{v_A}{v_B} \geq \frac{1}{\sqrt{2}}$, then the Dominance Region of Agent B, defined by $\bigcup_{t^*>0} I_B(t^*) \backslash I_A(t^*)$, is unbounded.

NOTE. This is because movement measured with respect to the L^1 metric is symmetric only along the axes, i.e. through a reflection over either axis or a rotation of $90°$. It is therefore sufficient to only consider cases where $c \geq 0$ and $d > 0$. We divide the proof for this theorem into two cases: $c = 0$ and $c > 0$. Starting with $c = 0$:

PROOF.

NOTE. This proof can be divided into four cases: $\gamma > 1$, $\gamma = 1$, $\frac{1}{\sqrt{2}} < \gamma < 1$, and $\gamma = \frac{1}{\sqrt{2}}$. When referring to the "region," the dominance region of Agent B (agent starting at the origin that is orange) is being described. Also, recall that γ is the velocity of Agent B (orange) divided by the velocity of Agent A (blue).

For the case where $\gamma > 1$, it is obvious that Agent B can escape because it is able to move faster that Agent A, thus rendering capture impossible. Thus, the region is unbounded for $\gamma > 1$.

For $\frac{1}{\sqrt{2}} < \gamma < 1$, we postulate the equations for the boundary of the dominance regions are approximated by oblique asymptotes of form $y = mx + b$ determined by equation $x^2 + (ax+b)^2 = \gamma^2(x^2 + 2|x||ax+b-d| + (ax+b-d)^2) + \varepsilon(x)$, with $\varepsilon(x)$ being an error term. As $x \longrightarrow \infty$ where $\varepsilon(x) \longrightarrow 0$ as $x \longrightarrow \infty$.

There are four equations determined for the range $\frac{1}{\sqrt{2}} < \gamma < 1$, with the quantity a being $\pm\frac{\gamma^2+\sqrt{2\gamma^2-1}}{1-\gamma^2}$ and $\pm\frac{\gamma^2-\sqrt{2\gamma^2-1}}{1-\gamma^2}$ and the quantity b being $\frac{d\gamma^2(1-a)}{a+\gamma^2(1-a)}$.

These four equations are as follows:

$$
\begin{cases}
(1)\ y = \dfrac{\gamma^2+\sqrt{2\gamma^2-1}}{1-\gamma^2}x + \dfrac{d\gamma^2\left(1-\frac{\gamma^2+\sqrt{2\gamma^2-1}}{1-\gamma^2}\right)}{\frac{\gamma^2+\sqrt{2\gamma^2-1}}{1-\gamma^2}+\gamma^2\left(1-\frac{\gamma^2+\sqrt{2\gamma^2-1}}{1-\gamma^2}\right)} \\[2ex]
(2)\ y = \dfrac{\gamma^2-\sqrt{2\gamma^2-1}}{1-\gamma^2}x + \dfrac{d\gamma^2\left(1-\frac{\gamma^2-\sqrt{2\gamma^2-1}}{1-\gamma^2}\right)}{\frac{\gamma^2-\sqrt{2\gamma^2-1}}{1-\gamma^2}+\gamma^2\left(1-\frac{\gamma^2-\sqrt{2\gamma^2-1}}{1-\gamma^2}\right)} \\[2ex]
(3)\ y = -\dfrac{\gamma^2+\sqrt{2\gamma^2-1}}{1-\gamma^2}x + \dfrac{d\gamma^2\left(1+\frac{\gamma^2+\sqrt{2\gamma^2-1}}{1-\gamma^2}\right)}{-\frac{\gamma^2+\sqrt{2\gamma^2-1}}{1-\gamma^2}+\gamma^2\left(1+\frac{\gamma^2+\sqrt{2\gamma^2-1}}{1-\gamma^2}\right)} \\[2ex]
(4)\ y = -\dfrac{\gamma^2-\sqrt{2\gamma^2-1}}{1-\gamma^2}x + \dfrac{d\gamma^2\left(1+\frac{\gamma^2-\sqrt{2\gamma^2-1}}{1-\gamma^2}\right)}{-\frac{\gamma^2-\sqrt{2\gamma^2-1}}{1-\gamma^2}+\gamma^2\left(1+\frac{\gamma^2-\sqrt{2\gamma^2-1}}{1-\gamma^2}\right)}
\end{cases}
$$

With these oblique asymptotes in place, it becomes clear the region of dominance is unbounded. By definition, these asymptotes will not intersect outside of the starting region. Thus, Agent B is able to escape along this open region for the range $\frac{1}{\sqrt{2}} < \gamma < 1$.

The equations for $\gamma = \frac{1}{\sqrt{2}}$ and $\gamma = 1$ still remain. The case $\gamma = 1$ is examined next.

For $\gamma = 1$, simplify equation 2.2: $2|x||y-d| - 2dy + d^2 = 0$.

Assume $|x|$ is very large.

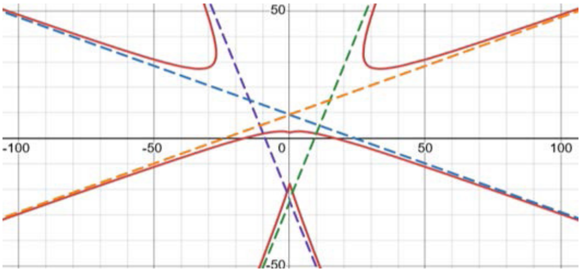


Figure 2.1
Oblique Asymptotes for $\gamma = \sqrt{0.6}$. (1): Green, (2): Orange, (3): Purple, (4): Blue

Assuming $x \gg 0$ and $y \geq d$, the equation becomes $-2x(d-y) - 2dy + d^2 = 0 \implies y(2x-2d) = 2xd - d^2 \implies y = d\frac{2x-d}{2x-2d}$, and, as $x \longrightarrow \infty$, $y \longrightarrow d$.

Assuming $x \ll 0$ and $y \geq d$, the equation becomes $-2x(y-d) - 2dy + d^2 = 0 \implies y(2x+2d) = 2xd - d^2 \implies y = d\frac{2x-d}{2x+2d}$, and, as $x \longrightarrow -\infty$, $y \longrightarrow d$.

Assuming $x \gg 0$ and $y < d$ results in $y = d\frac{2x+d}{2x+2d} = d\frac{2+\frac{d}{x}}{2+\frac{d}{x}}$. As $x \longrightarrow \infty$, $y \longrightarrow d$.

Assuming $x \ll 0$ and $y < d$ results in $y = d\frac{2x-d}{2x-2d}$. As $x \longrightarrow -\infty$, $y \longrightarrow d$.

This results in a single horizontal asymptote at $y = d$.

Now, assume y is very large.

Assuming $y \gg 0$ and $x > d$ results in $2x = \frac{2dy+d^2}{y-d}$. As $y \longrightarrow \infty$, $x \longrightarrow d$.

Assuming $y \ll 0$ and $x < -d$ results in $-2x = \frac{2dy+d^2}{y-d} = -d$. As $y \longrightarrow \infty$, $x \longrightarrow -d$.

Assuming $y \ll 0$ and $x > 0$ results in $2x = \frac{2dy+d^2}{y-d}$. $y \longrightarrow \infty$, $x \longrightarrow -d$. This is a contradiction.

Assuming $y \gg 0$ and $x < 0$ results in $2x = \frac{2dy+d^2}{y-d}$. As $y \longrightarrow \infty$, $x \longrightarrow d$. This is a contradiction.

This corresponds to two vertical asymptotes at $x = \pm d$. Therefore, Agent B's dominance region is not bounded for $\gamma = 1$. The results are plotted in Figure 2.5.

The only remaining case is $\gamma = \frac{1}{\sqrt{2}}$. The plot of the equation for this γ value will result in four parabolic branches.

Because of the term $|x|$, through symmetry, the solution for $x \geq 0$ is sufficient for both cases.

Simplifying yields the following equation:

$x^2 + y^2 - 2x|y-d| + 2dy - d^2 = 0$

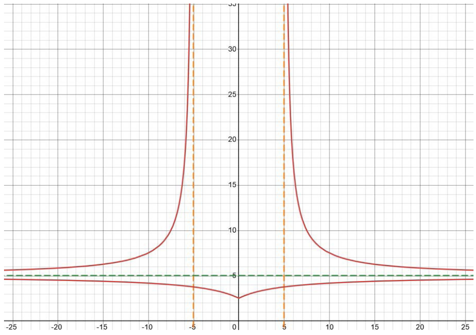


Figure 2.2
Plot for $\gamma = 1$ and $d = 5$ with horizontal and vertical asymptotes

We solve for $y \leq d$ because we are interested in the Dominance Region of Agent B:
$x^2 + y^2 - 2dy + 2dx + 2dy - d^2 = 0. \implies x^2 + y^2 - 2xd + 2xy + 2dy - d^2 \implies$
$y^2 + 2(d + x)y + x^2 - 2xd - d^2 = 0$

This is a quadratic. Solving for $y_{1,2}$:

$$y_{1,2} = \frac{-2(d+x) \pm \sqrt{4(x+d)^2 - 4(x^2 - 2xd - d^2)}}{2} = -(x + d) \pm \sqrt{x^2 + 2xd + d^2 - x^2 + 2xd + d^2}$$

$$y_{1,2} = -(x + d) \pm \sqrt{4xd + 2d^2}$$

These solutions for y define parabolic branches that describe the boundary of the Dominance Region of Agent B for $x \geq 0$ and $y \leq d$. It is therefore clear that this Dominance Region is unbounded. See Figure 2.3.

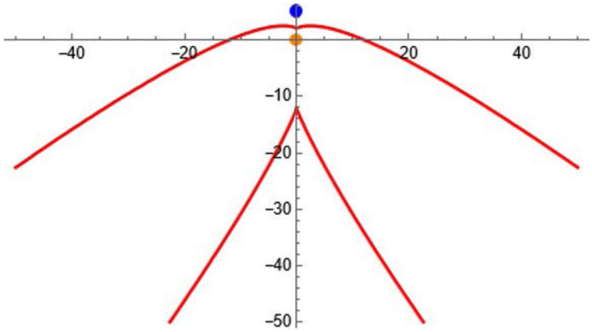


Figure 2.3
Parabolic Branches for $\gamma = \frac{1}{\sqrt{2}}$ and d $= 5$

Each of the proposed four cases have been proven to be unbounded regions for Agent B, thus, Agent B's dominance region is unbounded for $\gamma \geq \frac{1}{\sqrt{2}}$.$\square$

This was the case where Agent A starts at $(0, d)$ and Agent B starting at $(0, 0)$. If Agent A is instead started from (c, d), with $c \neq 0$, the proof will differ in every case. We now take $c > 0$:

PROOF.

NOTE. We take the same four cases for this proof as for the above proof.

For the case where $\gamma > 1$, it is obvious that Agent B can escape because it is able to move faster that Agent A, thus rendering capture impossible. Thus, the region is unbounded for $\gamma > 1$.

For $\frac{1}{\sqrt{2}} < \gamma < 1$, we postulate the equations for the boundary of the dominance regions are approximated by oblique asymptotes of form $y = mx + b$ determined by equation $x^2 + (ax + b)^2 = \gamma^2((x - c)^2 + 2|x - c||ax + b - d| + (ax + b - d)^2) + \varepsilon(x)$, with $\varepsilon(x)$ being an error term. As $x \longrightarrow \infty$, where $\varepsilon(x) \longrightarrow 0$ as $x \longrightarrow \infty$. The slope values, a, will be the same as before because a change in c will only shift the isochrones of Agent A. Only the value of b will change to include the new c term.

The quantity a is $\pm\frac{\gamma^2+\sqrt{2\gamma^2-1}}{1-\gamma^2}$ or $\pm\frac{\gamma^2-\sqrt{2\gamma^2-1}}{1-\gamma^2}$. The quantity b is now $\frac{d\gamma^2(1-a)}{a+\gamma^2(c+1-a(1+c))}$.

These four equations are as follows:

$$\begin{cases} (1)\ y = \dfrac{\gamma^2 + \sqrt{2\gamma^2 - 1}}{1 - \gamma^2}x + \dfrac{d\gamma^2(1 - \frac{\gamma^2+\sqrt{2\gamma^2-1}}{1-\gamma^2})}{\frac{\gamma^2+\sqrt{2\gamma^2-1}}{1-\gamma^2} + \gamma^2(c + 1 - \frac{\gamma^2+\sqrt{2\gamma^2-1}}{1-\gamma^2}(c+1))} \\ (2)\ y = \dfrac{\gamma^2 - \sqrt{2\gamma^2 - 1}}{1 - \gamma^2}x + \dfrac{d\gamma^2(1 - \frac{\gamma^2-\sqrt{2\gamma^2-1}}{1-\gamma^2})}{\frac{\gamma^2-\sqrt{2\gamma^2-1}}{1-\gamma^2} + \gamma^2(c + 1 - \frac{\gamma^2-\sqrt{2\gamma^2-1}}{1-\gamma^2}(c+1))} \\ (3)\ y = -\dfrac{\gamma^2 + \sqrt{2\gamma^2 - 1}}{1 - \gamma^2}x + \dfrac{d\gamma^2(1 + \frac{\gamma^2+\sqrt{2\gamma^2-1}}{1-\gamma^2})}{-\frac{\gamma^2+\sqrt{2\gamma^2-1}}{1-\gamma^2} + \gamma^2(c + 1 + \frac{\gamma^2+\sqrt{2\gamma^2-1}}{1-\gamma^2}(c+1))} \\ (4)\ y = -\dfrac{\gamma^2 - \sqrt{2\gamma^2 - 1}}{1 - \gamma^2}x + \dfrac{d\gamma^2(1 + \frac{\gamma^2-\sqrt{2\gamma^2-1}}{1-\gamma^2})}{-\frac{\gamma^2-\sqrt{2\gamma^2-1}}{1-\gamma^2} + \gamma^2(c + 1 + \frac{\gamma^2-\sqrt{2\gamma^2-1}}{1-\gamma^2}(c+1))} \end{cases}$$

With these oblique asymptotes in place, it becomes clear the region of dominance is unbounded. Thus, Agent B is able to escape along this open region for the range $\frac{1}{\sqrt{2}} < \gamma < 1$.

The equations for $\gamma = \frac{1}{\sqrt{2}}$ and $\gamma = 1$ still remain. The case $\gamma = 1$ is examined next.

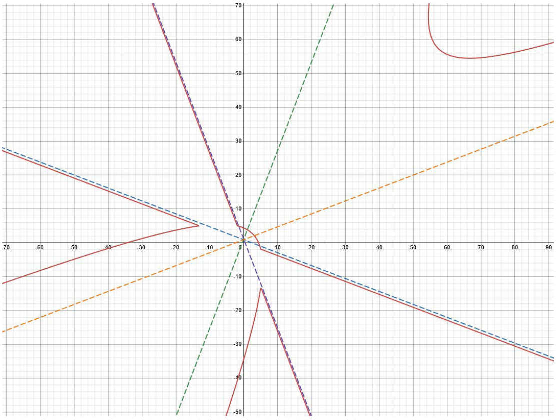


Figure 2.4

Oblique Asymptotes for $\gamma = \sqrt{0.6}$ and Agent A starting location at (5,5).
(1): Green, (2): Orange, (3): Purple, (4): Blue

For $\gamma = 1$, simplify equation 2.2: $-2cx + c^2 + 2|x - c||y - d| - 2dy + d^2 = 0$.

Assume $|x|$ is very large.

Assuming $x \gg 0$ and $y \geq d$, the equation becomes $-2x(d-y) - 2cx - 2y(c+d) + c^2 + d^2 = 0 \implies y(2x - 2c - 2d) = 2x(c+d) - c^2 - d^2 \implies y = \frac{2x(c+d)-c^2-d^2}{2x-2c-2d}$, and, as $x \longrightarrow \infty$, $y \longrightarrow c + d$.

Assuming $x \ll 0$ and $y \geq d$, the equation becomes $-2x(y-d) - 2cx - 2y(c+d) - c^2 + d^2 = 0 \implies y(-2x - 2c - 2d) = 2x(c-d) - c^2 - d^2 \implies y = -\frac{2x(c-d)-c^2-d^2}{2x+2c+2d}$, and, as $x \longrightarrow -\infty$, $y \longrightarrow d - c$. This is a contradiction for $c \neq 0$

Assuming $x \gg 0$ and $y \leq d$ results in $y = \frac{2x(c+d)-c^2-d^2}{2x-2d}$. As $x \longrightarrow \infty$, $y \longrightarrow d - c$.

Assuming $x \ll 0$ and $y \leq d$ results in $y = \frac{2x(c+d)-c^2-d^2}{2x-2d}$. As $x \longrightarrow -\infty$, $y \longrightarrow c + d$. This is a contradiction for $c \neq 0$

This results in two horizontal asymptotes at $y = c + d$ and $y = d - c$.

Now, assume y is very large.

Assuming $y \gg 0$ and $x > c$ results in $x = \frac{2y(c+d)-c^2-d^2}{2y-2c-2d}$. As $y \longrightarrow \infty$, $x \longrightarrow c + d$.

Assuming $y \ll 0$ and $x > c$ results in $x = -\frac{2y(-c+d)-c^2-d^2}{2y+2c+2d}$. As $y \longrightarrow \infty$, $x \longrightarrow c - d$. This is a contradiction.

Assuming $y \gg 0$ and $x < c$ results in $x = \frac{2y(-c+d)-c^2-d^2}{2y+2c+2d}$. $y \longrightarrow \infty$, $x \longrightarrow c - d$.

Assuming $y \gg 0$ and $x < c$ results in $x = \frac{2y(c+d)-c^2-d^2}{2y-2c-2d}$. As $y \longrightarrow \infty$, $x \longrightarrow c + d$. This is a contradiction.

This corresponds to two vertical asymptotes at $x = c + d$ and $x = c - d$. Therefore, Agent B's dominance region is not bounded for $\gamma = 1$. The results are plotted in Figure 2.5.

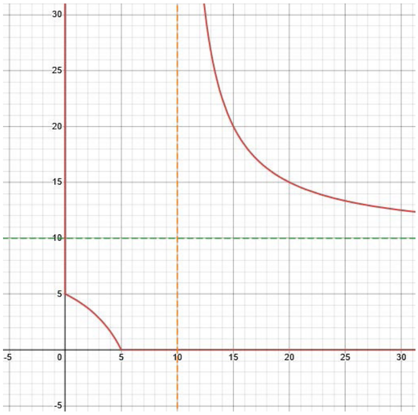


Figure 2.5
Plot for γ = 1 and c = d = 5 with horizontal and vertical asymptotes

The only remaining case is $\gamma = \frac{1}{\sqrt{2}}$. The plot of the equation for this γ value will result in at most four parabolic branches.

Simplifying yields the following equation:

$x^2 + y^2 - 2|x - c||y - d| + 2cx + 2dy - c^2 - d^2 = 0$

Assume $y \leq d$ because the Dominance Region of Agent B is of interest. We start with the case with $x > c$:

$x^2 + y^2 + 2xy - 2dx + 2cd - 2cy + 2cx + 2dx + 2dy - c^2 - d^2 = 0. \implies y^2 + 2(x - c + d)y + x^2 - 2(c - d)x + 2cd - c^2 - d^2 = 0$

This is a quadratic. Solving for $y_{1,2}$:

$y_{1,2} = \frac{-2(x-c+d)\pm\sqrt{4(x-c+d)^2-4(x^2-2(c-d)x+2cd-c^2-d^2)}}{2} =$
$-(x - c + d) \pm \sqrt{x^2 + 2(d - c)x + c^2 - 2cd + d^2 - x^2 + 2(d - c)x - 2cd + c^2 + d^2}$

$y_{1,2} = -(x - c + d) \pm \sqrt{4(d - c)x - 4cd + 2c^2 + 2d^2}$

Now we solve for when $x < c$:

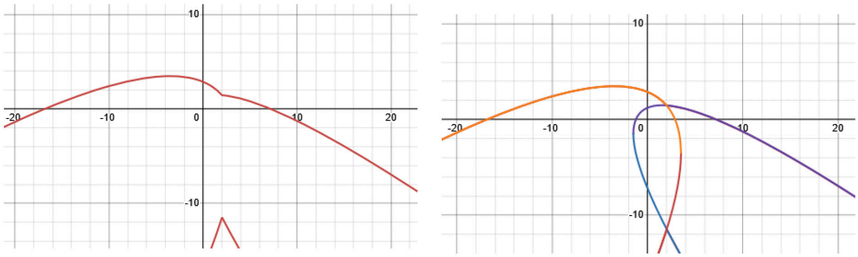
$x^2 + y^2 + 2cx - 2xy - 2dx - 2cd + 2cy + 2dx + 2dy - c^2 - d^2 = 0. \implies y^2 + 2(c + d - x)y + x^2 + 2(c + d)x - 2cd - c^2 - d^2 = 0$

This is a quadratic. Solving for $y_{1,2}$:

$y_{1,2} = \frac{-2(c+d-x)\pm\sqrt{4(c+d-x)^2-4(x^2+2(c+d)x-2cd-c^2-d^2)}}{2} =$
$x - (c + d) \pm \sqrt{c^2 + 2cd + d^2 - 2(c + d)x + x^2 - x^2 - 2(c + d)x + 2cd + c^2 + d^2}$

$y_{1,2} = x - (c + d) \pm \sqrt{-4(c + d)x + 4cd + 2c^2 + 2d^2}$

These solutions for y define parabolic branches that describe the boundary of the Dominance Region of Agent B $y \leq d$. It is therefore clear that this Dominance Region is unbounded. See Figure 2.6b.



(A) Dominance Region of Agent B with Agent A starting at $(2,5)$ and $\gamma = \frac{1}{\sqrt{2}}$.

(B) Parabolic Branches Fit to the example in Figure 2.6a.

Figure 2.6
Example of Parabolic Branches fit to Dominance Region.

Each of the proposed four cases have been proven to be unbounded regions for Agent B, thus, Agent B's dominance region is unbounded for $\gamma \geq \frac{1}{\sqrt{2}}$ and Agent A starting at (c,d).□

Motion in Presence of Barrier

The effects of a finite straight-line barrier were studied in this exercise not to prove a theorem surrounding its effects, but to motivate further study through numerical examples. There are too many analytical and algebraic cases to make any broad conclusions about interactions of an agent whose movement is measured with respect to the L^1 metric at the moment. This is a first attempt to see how such a barrier affects an agent whose movement is measured with respect to the L^1 metric.

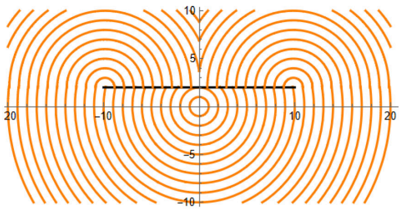
The Barrier's Definition and How It Affects the Agent's Isochrones

Introduce a barrier understood as a finite, straight line. To model this, the barrier is treated as an impassible region. Either target must move to the end points of the barriers and have its isochrones permeate out from there. The methods for this differ between the two means of describing movement.

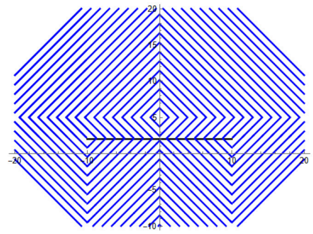
DEFINITION 7 (Line of Sight). *The line of sight (LOS) is a linear curve of the form $y = mx + l$ that connects the observer at (x_a, y_a) and the observed at (x_b, y_b).*

The L^2 case is discussed extensively in the paper Oyler et al. 2016, but it depends largely on the line of sight (LOS) of the agent to the edges of the barrier; a simple case is shown in Figure 3.1a.The region up to the lines of sight and barrier are treated the same as with no barrier. Past these lines, the isochrones are plotted as if the agent started from the edge of the barrier, with a difference in the starting time, t, to account for the time it takes to move to the edge of the barrier. The line where the isochrones intersect above the barrier (the y-axis above the barrier in this case) represents the set of points at which it would take the same time to reach traveling around the barrier either direction, when starting from the agent's initial location.

REMARK (Axis of Motion). *In the L^1 metric, the distance between points (x_1,y_1) and (x_2,y_2) can be decomposed as $d_1((x_1,y_1),(x_2,y_2)) = d_1((x_1,y_1),(x_1,y_2)) + d_1((x_1,y_2),(x_2,y_2)) = d_1((x_1,y_1),(x_2,y_1)) + d_1((x_2,y_1),(x_2,y_2))$. Hence, the movement of an agent described with* *respect to the L^1 metric may as well be decomposed into a horizontal and vertical component with their sums equating to the whole distance traveled. This observation gives rise to the notion of axes of motion: One axis of motion denoting a horizontal component and the other a vertical component.*



(A) Simple case for Agent B's isochrones around a barrier.



(B) Simple case for Agent A's isochrones around a barrier (single AoM blocked).

Figure 3.1
The barrier's definition and how it affects the agent's isochrones

The lines along which the isochrones are affected is along one of these axes of motion. As in **Figure 3.1b**, the isochrones change along vertical lines on either side of the barrier as opposed to the line of sight with the center of the barrier as in the L^2 case.

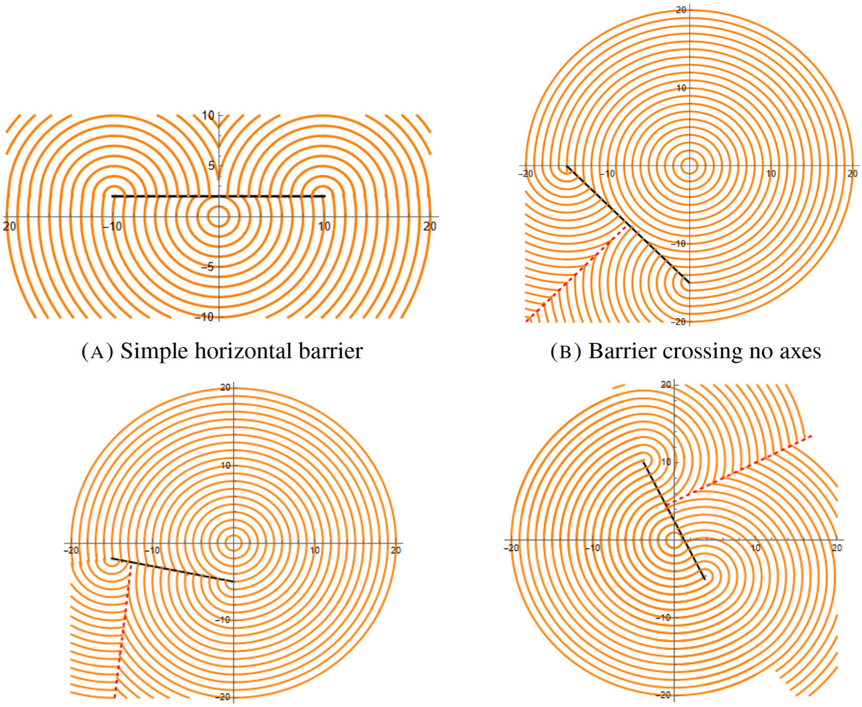
Two terms will be used to describe the change that the barrier causes for each agent:

REMARK (Global Change). *Global change is defined as a change to the isochrones that does not correct itself with time, i.e, in **Figures 3.3a** and **3.3d**, the shape never returns to the original diamond shape.*

REMARK (Local Change). *Local change is defined as a change to the isochrones that is limited to a region directly behind the barrier (with relation to the agent). Such change is only possible for an agent whose movement is measured with respect to the L^1 norm with a barrier which does not block one of its axes of motion.*

Barrier-Isochrone Behaviors with Respect to the L^2 Norm

An agent whose movement is measured with respect to the L^2 metric will always have change to its isochrones with a barrier present, as the L^2 agent can no longer choose a straight line curve to a point past the barrier. Some cases are examined in **Figure 3.2**. Observe that there is global change to each of the isochrones past the barrier.



(A) Simple horizontal barrier

(B) Barrier crossing no axes

(C) Barrier crossing no axes

(D) Barrier crossing both axes

Figure 3.2

Various barrier positions and the effect on the isochrones of an L^2 agent.

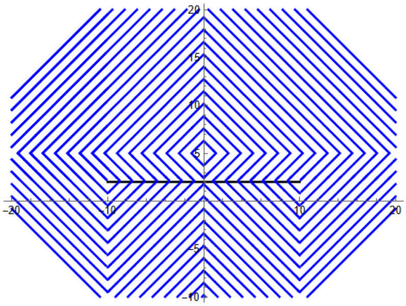
Numerical Exploration for the L1 Metric

Effects of Barrier on Motion Measured with Respect to the L1 Metric

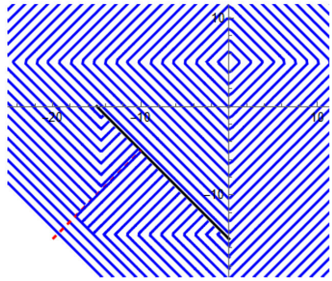
The addition of a finite, straight-line barrier affects an agent's, affects the isochrones and the resulting dominance region either locally or globally depending on its position with respect to the agent's axis of motion. This is represented in three cases: one axis of motion (AoM) blocked, both AoM blocked, and no AoM blocked. The case with single AoM and both AoM being blocked results in global change of the dominance regions, whereas the case with no AoM blocked only results in local change.

NOTE. We explore three separate cases: one AoM blocked, two AoM blocked, and no AoM blocked. These cases are examined for the effects on the isochrones and the resulting dominance regions. To start, the case with one axis of motion blocked:

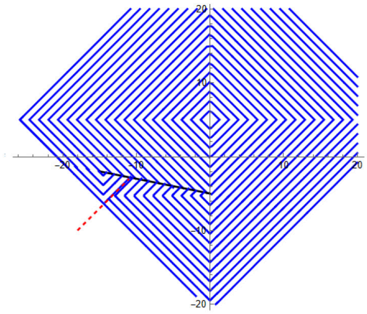
Effects on the Isochrones



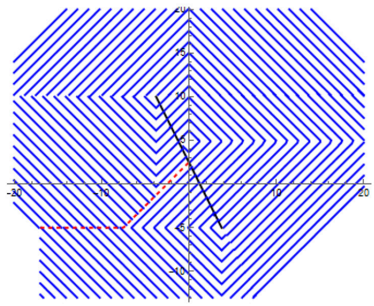
(A) Single AoM Blocked



(B) No AoM Blocked



(C) No AoM Blocked



(D) Both AoM Blocked

Figure 3.3
Various barrier positions and the effect on the isochrones of Agent A.

For **Figure 3.3a**, a single AoM is blocked, causing global change [3.1] along the negative y-axis. Its movement along the x-axis is unchanged, however. Observe that only the section defined by the vertical tangents on either side of the barrier is affected. The change to the agent's isochrones is still global because it will take longer go around the barrier and reach any point behind it, as it cannot move through the barrier. This global change is only present along the axis of motion that is blocked and is limited by the length of the barrier. Thus, there is global change to the agent's isochrones if one axis of motion is blocked.

Next, the effects of a barrier blocking no AoM is examined.

For **Figures 3.3b** and **3.3c**, there is local change [3.1] surrounding the barrier, but the change 'corrects' itself with time, i.e, the isochrones return to their original shape once the agent's traveled distance is greater than the vertical or horizontal legs of the barrier (if the barrier were treated as a hypotenuse). This is because Agent A may simply choose a path that goes completely around the barrier along either of its axes of motion. Thus, there is only local change for an agent's isochrones when the barrier blocks no AoM.

There is one more case, where both of the agent's axes of motion are obstructed.

In **Figure 3.3d**, there is global change along both axes of motion. This corresponds to change for both the vertical and horizontal components of the agent's movement past the barrier because it has to move around the barrier. Thus, there is global change along both AoM when the barrier blocks both of these AoM.

Effects on the Dominance Region

To determine how the dominance regions were affected by the barrier, the Dominance Regions were found in sections, with the sections where the agent is treated as starting from the edge of the barrier having a distance equal to that it would have to travel shifting it directly away from the other agent. This is because when parameterizing the equation, we removed time from the isochrone equations, thus we must treat the distance traveled (constant velocity times time) as the time difference in this case.

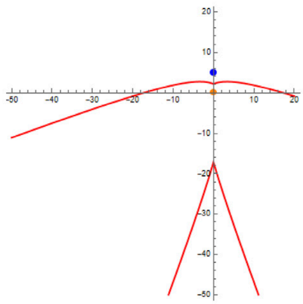
NOTE. All examples are graphed with a $\gamma 2 = 0.6$. The dominance region is affected in a very similar way to the Agent A's isochrones.

In **Figure 3.4b**, when compared to the base case in Figure 3.4a, there is visible global change in the dominance region only in the vertical direction of the barrier because it affects only that AoM. The change affects both agents: the region above the barrier is effectively deleted, adding to Agent A's dominance region. The region directly below the barrier is shifted downwards, corresponding to global change in the dominance region of both Agent A (Blue) and Agent B (Orange) as a consequence. Thus, there is global change to the dominance region along the axis of motion blocked by the barrier.

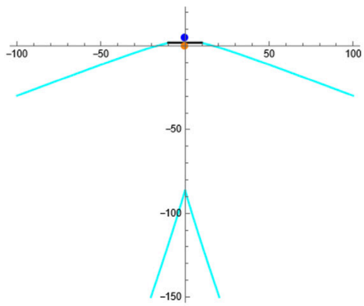
Next, the case when both AoM are blocked will be examined.

Figure 3.4c shows the case with both axes of motion blocked. When compared to the base case in Figure 3.4a, it is obvious that there is global change in both directions from the barrier. Agent A (Blue) benefits from the barrier slightly on the region just to the right of the barrier and the angled region below the barrier is shifted slightly up and to the right. In every other region, Agent B's (Orange) dominance region is expanded, however, enabling more options for paths used to escape contained within the unbounded region. Thus, there is global change to the dominance region along both axes of motion of Agent A.

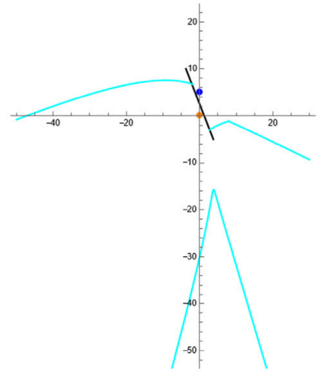
Lastly, the case in which the barrier blocks no axes of motion is examined.



(A) Base case with no barrier for B and C



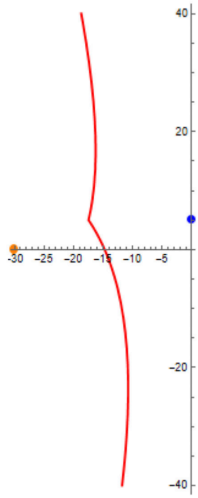
(B) One AoM blocked for Pursuer(blue)



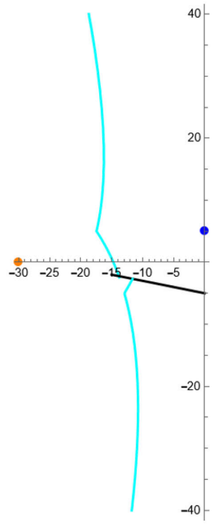
(C) Both AoM blocked for Pursuer

Figure 3.4
The dominance regions of a few of the cases from section 4.2

Figure 3.5b shows the case with no AoM blocked, and when compared against the base case (**Figure 3.5a**), it should be observed that there is only local change in the region directly behind the barrier with relation to Agent A. This corresponds to a slight increase in the dominance region of Agent B (Orange) and a resulting decrease in the region for Agent A (Blue). Thus, there is only local change in the dominance region when the barrier does not block Agent A's AoM.



(A) Base case with no Barrier for B



(B) No AoM blocked for Pursuer

Figure 3.5
Base case and Barrier case for no AoM blocked

Conclusion

This paper entailed the exploration of escape and pursuit games, as well as the original work on limiting one agent to movement measured with respect to the L^1 metric and studying the effects of a barrier on such interactions. It was found that barriers affect an agent with movement measured with respect to the L^1 metric in a very different way than one with movement described with respect to the Euclidean (L^2) metric. The most significant finding was the L^1-L^2 Interaction Theorem, stating that for a velocity ratio of $\gamma = \frac{1}{\sqrt{2}} \approx 70.7\%$, a target is still able to escape if moving unrestricted (with respect to the L^2 metric). The second was a numerical exploration describing the barrier's effect on the isochrones and resulting dominance regions. This finding aligns with the first theorem. It provides insight into various applications and hints at a formal theorem.

Future Outlook

The current work lays the foundation for further work. Velocity was assumed to be constant and turn time/radius was not considered, but acceleration, both linear and angular, could be added in order to add complexity to the model and make it more applicable to realworld phenomena. The barrier is a finite, straight line, but the shape of the barrier could be generalized to a polygon or even a circle/ellipse. The movement could also be generalized to three dimensions, however this would require a lot of restructuring and would be rather difficult to visualize 3-dimensional dominance regions and isochrones. Additionally, there is more work to be done to expand the exercise from this paper into a formal theorem, there are many possible algebraic possibilities for placing the barrier.

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Oyler, Dave W., Pierre T. Kabamba and Anouck R. Girard (2016). 'Pursuit–evasion games in the presence of obstacles'. In: Automatica 65, pp. 1–11. ISSN: 0005-1098. DOI: https://doi.org/10.1016/j.automatica.2015.11.018. URL: https://www. sciencedirect.com/science/article/pii/S0005109815004823.