Random transceiver networks

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Abstract

Consider randomly scattered radio transceivers in $\mathbb{R}^d$, each of which can transmit signals to all transceivers in a given randomly chosen region about itself. If a signal is retransmitted by every transceiver that receives it, under what circumstances will a signal propagate to a large distance from its starting point? Put more formally, place points $\{x_i\}$ in $\mathbb{R}^d$ according to a Poisson process with intensity 1. Then, independently for each x_i, choose a bounded region A_{x_i} from some fixed distribution and let $\mathcal{G}$ be the random directed graph with vertex set $\{x_i\}$ and edges $x_i\vec{x}_j$ whenever $x_j \in x_i + A_{x_i}$. We show that for any $\eta > 0$, $\mathcal{G}$ will almost surely have an infinite directed path, provided the expected number of transceivers that can receive a signal directly from x_i is at least $1 + \eta$, and the regions $x_i + A_{x_i}$ do not overlap too much (in a sense that we shall make precise).

One example where these conditions hold, and so gives rise to percolation, is in $\mathbb{R}^d$, with each A_{x_i} a ball of volume $1 + \eta$ centred at x_i, where $\eta \to 0$ as $d \to \infty$. Another example is in two dimensions, where the A_{x_i} are sectors of angle $\varepsilon\theta$ and area $1 + \eta$, uniformly randomly oriented within a fixed angle $(1 + \varepsilon)\theta$. In this case we can let $\eta \to 0$ as $\varepsilon \to 0$ and still obtain percolation. The result is already known for the annulus, i.e., that the critical area tends to one as the ratio of the radii tends to one, while it is known to be false for the square (l_∞) annulus. Our results show that it does however hold for the randomly oriented square annulus.

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1 Introduction

We consider an ad-hoc wireless network consisting of many transceivers randomly distributed over a large region, some or all of which have sensors that record some local data. Each transceiver can transmit its data to some nearby region. All the recipients of this information then retransmit it. Under some conditions, (e.g., enough power and enough transceivers) the information can propagate a large distance. In particular, if we listen at the boundary of the large region we may still pick up information originating from sensors in the centre of the region. We wish to prove some bounds on the power and number of transceivers needed for this to occur.

A very natural way to model this is to suppose that the transceivers are distributed according to a Poisson process in some region, for example a square, and that the transceivers are omni-directional, i.e., that any transceiver within a disc about the transceiver can receive the information (where the radius of the disc may depend on the power of the transceiver). A limiting version of this problem is to suppose that the transceivers are distributed according to a Poisson process in the entire plane and ask for the existence of an infinite component; i.e., whether percolation occurs. In this case it is known that we need the power to be such that, on average, a transceiver can broadcast to approximately 4.512 other transceivers (see [12] for numerical simulations and [4] for a semi-rigourous result). It is natural to ask whether we can do better if we transmit to some other region, in particular whether directional transmission can help. In this paper we show that, given any reasonable model of directional transceiver, we can do better. Moreover this can be achieved with the transceivers randomly oriented; in particular the transceivers do not require any global knowledge of the arrangement.

To model this setup, fix a probability distribution $\mathcal{D}$ on measurable regions $A \subseteq \mathbb{R}^d \setminus \{0\}$. Construct a random digraph $\mathcal{G}$ by placing points $\{x_i\}$ in $\mathbb{R}^d$ according to a Poisson process with intensity 1. Choose independently for each x_i, regions A_{x_i} according to the distribution $\mathcal{D}$. Let the vertices of $\mathcal{G}$ be the x_i, and let the edges $\vec{x_i x_j}$ lie in $\mathcal{G}$ when $x_j \in A(x_i) := x_i + A_{x_i}$. The points x_i represent the locations of our transceivers, and the sets $A(x_i)$ represent the region in which the signal strength of x_i is sufficiently strong to be received by another transceiver. We assume that the *reception* of a signal is not directional, so all transceivers within $A(x_i)$ can receive data from x_i. Individual transceivers may be strongly directional, so $A(x_i)$ may be highly non-symmetric. Indeed, even the distribution $\mathcal{D}$ may be very irregular, due to variation in power, or partial failure of the transceivers, or even global bias in the direction of transmission. We wish to know under what circumstances $\mathcal{G}$ has an infinite directed path.

We shall show that for an infinite directed path to exist we only need the expected volume of A_{x_i} to be slightly more than 1, provided there is not much overlap in the regions A_{x_i} (in a sense that will be made precise below), and the distribution $\mathcal{D}$ satisfies some mild boundedness conditions.

Throughout this paper we shall denote by $|S|$ the standard Lebesgue measure of S in $\mathbb{R}^d$. We shall fix a distribution $\mathcal{D}$ on regions, and define two distributions derived from $\mathcal{D}$. Let $\mathcal{D}_0$ be the distribution of the number N of points of our Poisson process in A where A is distributed according to $\mathcal{D}$. Note that if all the volumes $|A|$ are the same, then $\mathcal{D}_0$ is a Poisson distribution with mean $|A|$. Let $\mathcal{D}_c$ be the probability distribution on $\mathbb{R}^d$ given by

$$\mathbb{P}_{\mathcal{D}_c}(S) = \mathbb{E}(|S \cap A|)/\mathbb{E}(|A|) \tag{1}$$

where A is distributed according to $\mathcal{D}$. In other words, $\mathcal{D}_c$ is the unconditioned probability distribution of the location of a 'typical' neighbour of the point 0.

We now introduce several parameters that describe $\mathcal{D}$. The most important of these, and most difficult to describe, is the parameter δ which will indicate the amount of overlap we get between the regions $A(x_i)$.

We require that for any x_1, x_2, if we fix $A(x_1)$ and $A(x_2)$, then the probability of a randomly chosen neighbour of a randomly chosen neighbour of x_1 being a neighbour of x_2 is at most δ. In addition, we also need the same result if we, rather artificially, replace A_{x_1} by $-A_{x_1}$. More formally, let the sets A and A' be distributed according to $\mathcal{D}$. Fixing A, A', and $z \in \mathbb{R}^d$, let X be a random point uniformly distributed in A and let Y be an independent random variable with distribution $\mathcal{D}_c$. Then define δ by

$$\delta = \operatorname*{ess\,sup}_{A,A'} \sup_z \max\{\mathbb{P}(X + Y \in A' + z), \mathbb{P}(-X + Y \in A' + z)\}. \tag{2}$$

(If $A(x_1) = A + x_1$, $A(x_2) = A' + x_2$, and $z = x_2 - x_1$ then we obtain the previous description.) Note that if $\mathcal{D}$ is centrally symmetric, then the $\mathbb{P}(-X + Y \in A' + z)$ term above is unnecessary since we can replace A by $-A$. Note also that by changing $\mathcal{D}$ on a set of measure zero, we may assume 'ess sup' is the same as 'sup'.

To illustrate this definition, consider the distribution $\mathcal{D}$ which gives a fixed region A with probability 1. Then δ is given more simply by

$$\delta = |A|^{-2} \sup_z \int_{A-z} |(A + x) \cap A| \, dx. \tag{3}$$

(Write $x = X \mp z$, $A' = A$, and note that X and Y are now both uniformly distributed in A.) For example, suppose A is a d-dimensional sphere with radius r centred at the origin in $\mathbb{R}^d$. Then $(A + x) \cap A$ is contained in a sphere of radius $r\sqrt{1 - \alpha^2/4}$ about the point $x/2$, where $\alpha = \|x\|$. Hence $|(A + x) \cap A| \le (1 - \alpha^2/4)^{d/2}|A|$. Since $|(A + x) \cap A|$ is decreasing for increasing $\|x\|$, the supremum in (3) occurs when $z = 0$. Thus

$$\delta \le \int_0^1 (1 - \alpha^2/4)^{d/2}(d\,\alpha^{d-1}) \, d\alpha.$$

Substituting $\alpha = e^{-z}$ and using $1 - \alpha^2/4 = 1 - \frac{1}{4}e^{-2z} \le \frac{3}{4} + \frac{1}{2}z \le 0.75e^{2z/3}$ gives

$$\delta \le \int_0^\infty (0.75e^{2z/3})^{d/2} d\, e^{-z(d-1)} e^{-z} dz = (0.75)^{d/2} \int_0^\infty d\, e^{-(2d/3)z} \, dz = 1.5(0.75)^{d/2}, \tag{4}$$

so δ decreases exponentially with d. (By calculating (3) exactly, one can show that $\delta = \frac{3}{2}\big(\int_0^{\pi/3}\sin^d\theta\,d\theta\big)\big/\big(\int_0^{\pi/2}\sin^d\theta\,d\theta\big)$. Using this one can see that the exponential rate of decay given by (4) is sharp, although, for example, $\delta = 1 - (\sqrt{27}/4\pi) \approx 0.5865$ when $d = 2$, while (4) gives an estimate larger than 1.)

Another example is in 2 dimensions with A a randomly oriented thin sector of a disc. This example models the case when the transceivers are highly directional, but not all oriented in the same direction. To be precise, assume each sector has angle $\varepsilon\theta$, and the sectors are oriented uniformly over some fixed angle θ, $0 < \theta \leq 2\pi$, so that all the sectors lie within an angle of $(1+\varepsilon)\theta$ (or 2π if $(1+\varepsilon)\theta \geq 2\pi$). Then the probability distribution $\mathcal{D}_c$ has density at most $\varepsilon/|A|$ at any point. It is then clear that $\pm X + Y$ has probability density at most $\varepsilon/|A|$ anywhere, so $\delta = \sup_{z,A,A'} \mathbb{P}(\pm X + Y \in A' + z) \leq (\varepsilon/|A|)|A'| = \varepsilon$. More generally, we have:

Lemma 1. *If almost all volumes $|A|$ given by $\mathcal{D}$ are the same and for almost all fixed $z \in \mathbb{R}^d$ the probability that $z \in A$ (A distributed according to $\mathcal{D}$) is at most ε, then $\delta \leq \varepsilon$.*

Note that in our example, for δ to be small it is necessary that ε be small. It is not sufficient that θ be small. Indeed, reducing θ has little effect, since by applying a suitable area-preserving linear transformation, a small θ is the same as taking θ large and replacing the thin sectors by appropriate triangular shaped regions. It is then not hard to see that if ε fails to be small, δ will also not be small. Using Theorem 5, one can also show that in this case the area of the sectors A must be significantly larger than 1 if an infinite directed path is to exist.

One should also note that in Lemma 1, the condition that $|A|$ is constant is required. For example, consider the case when $A = \emptyset$ with probability $1-\varepsilon$ and A is a disc of area C/ε with probability ε. The probability that $z \in A$ is always at most ε, however δ is the same as if A were a disc of area C with probability 1, and is therefore independent of ε. Indeed, the behaviour of these two models is essentially the same since one can remove the points of the Poisson process in the first model where $A_{x_i} = \emptyset$ without altering the percolation properties. But then one obtains the second model scaled up by a factor $1/\varepsilon$ in area. Moreover, the second model percolates only for C larger than some critical value which is known to be at least 2 (see [11]), and experimental evidence suggests is about 4.512 (see [12]), hence in both models we need significantly more than 1 neighbour on average to ensure infinite directed paths.

The other parameters we need are somewhat easier to describe. Define η so that $1 + \eta$ is the average number of neighbours of a point in $\mathcal{G}$, which is also the average volume $\mathbb{E}|A|$. Define σ^2 to be the variance of the number of neighbours of a point. We shall see below that $\sigma^2 = \mathbb{E}|A| + \mathrm{Var}|A|$, so $\sigma^2 \geq 1 + \eta$ with equality if and only if the volume $|A|$ is constant. We shall assume that all sets A given by our distribution $\mathcal{D}$ lie in a ball of radius r_0 about 0. We also assume the root mean square distance of a neighbour is at

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least $r_m > 0$ in any direction, i.e.,

$$r_m^2 \leq \mathbb{E}((Y \cdot u)^2) \qquad \text{for any unit vector } u, \tag{5}$$

where Y is distributed according to $\mathcal{D}_c$. We can now state the main result.

Theorem 2. *There is an absolute constant $c > 0$ such that if $\eta \leq 1$ and*

$$\delta < c r_m^9 r_0^{-9} \eta^{16} \sigma^{-32}$$

then $\mathcal{G}$ almost surely has an infinite directed path.

From Theorem 2 we can deduce two immediate corollaries.

Corollary 3. *Assume that $\mathcal{G}$ consists of the points of a Poisson process with intensity 1 in $\mathbb{R}^d$, $d \geq 2$, and each point is joined to all other points within a ball of volume $1 + \eta$. Then there exists a constant $c > 0$, independent of d, such that if $\eta > c(0.9911)^d$ then $\mathcal{G}$ almost surely has an infinite component.*

Proof. The distribution $\mathcal{D}$ gives the ball with volume $1+\eta$ with probability 1. An infinite directed path is an infinite component since $\vec{xy} \in \mathcal{G}$ if and only if $\vec{yx} \in \mathcal{G}$. From the above discussion, $\delta \leq 1.5(0.75)^{d/2}$. But r_0/r_m is bounded by a polynomial in d (in fact $r_0/r_m = \sqrt{d+2}$), and $\sigma^2 = 1 + \eta$ is bounded when $\eta \leq 1$. The result follows for large d since $0.9911^{16} > \sqrt{0.75}$. Note that for any $d \geq 2$ there does exist some η such that $\mathcal{G}$ has an infinite component. To see this, divide up $\mathbb{R}^d$ into cubes of side length L and identify each cube with a point in $\mathbb{Z}^d$ in the obvious manner. Define a site percolation on $\mathbb{Z}^d$ by declaring a site open if the corresponding cube contains a point of the Poisson process. Assume η is such that the ball of volume $1 + \eta$ has radius at least $L\sqrt{(d-1)+4}$. Then any pair of points in adjacent cubes are joined. Thus if the site percolation on $\mathbb{Z}^d$ has an infinite open component, then $\mathcal{G}$ must also have an infinite component. Thus provided we choose L such that $1 - e^{-L^d}$ is more than the critical probability of site percolation in $\mathbb{Z}^d$ (which is known to be strictly less than 1), then $\mathcal{G}$ almost surely has an infinite component. Hence by increasing c if necessary, the result holds for all $d \geq 2$. $\qquad\square$

Corollary 4. *Assume that $\mathcal{G}$ consists of the points of a Poisson process with intensity 1 in $\mathbb{R}^2$, and each point is joined to all other points within a sector of a disc of area $1 + \eta$ and angle $\varepsilon\theta$, randomly oriented over a fixed angle of θ. Then there exists a constant $c > 0$, independent of both θ and ε, such that if $\eta > c\varepsilon^{1/16}$ then $\mathcal{G}$ almost surely has an infinite directed path.*

Proof. From the above discussion we have $\delta \leq \varepsilon$. Assume first that ε is bounded and $\eta \leq 1$, so $\sigma^2 \leq 2$. For θ bounded away from zero, r_0/r_m is bounded and the result follows from Theorem 2 for some c. As $\theta \to 0$, $r_0/r_m \to \infty$, but if we apply an area-preserving linear transformation of $\mathbb{R}^2$, we can make all the sensor regions approximately triangular

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with r_0/r_m bounded. Thus one can choose c independently of θ for all $\theta > 0$. Since increasing the radius of the sectors only makes percolation more likely, the result follows for all η when ε is sufficiently small ($c\varepsilon^{1/16} < 1$).

For larger ε note that if $\mathcal{G}$ has an infinite directed path for $\varepsilon = \varepsilon_0$ with $\eta = 1$, then $\mathcal{G}$ still has an infinite directed path if we increase ε (and hence η) while keeping the radius of the sectors the same. Hence by increasing c if necessary, the result follows for all $\varepsilon \in (0, 2]$. For $\varepsilon > 2$, the sectors always contain a fixed sector of angle $(\varepsilon - 1)\theta$, and thus a triangle of area some constant fraction of $1 + \eta$. By a suitable linear transformation, we can make these triangles equilateral, so percolation will occur once this area is above some absolute constant, independently of the shape of the original triangle. Thus, by increasing c again if necessary, we obtain the result for all ε. $\qquad\square$

For another model take A to be a fixed annulus of area $1 + \eta$ and inner and outer radii $r(1 - \varepsilon)$ and r respectively where $r = r(\eta, \varepsilon)$ is determined by the requirement that $|A| = \pi r^2 (2 - \varepsilon)\varepsilon = 1 + \eta$. In this case, transceivers transmit data to other transceivers only if they are at distance between $r(1 - \varepsilon)$ and r. For this model it is already known that for any $\eta > 0$ we obtain percolation for sufficiently small ε (see [3] and [7]). In [3] it is also shown that this result fails for a "square" annulus, defined by $A = \{(x, y) : r(1 - \varepsilon) \le \max(|x|, |y|) \le r\}$. Indeed, we require $\eta > 0.014$ for all $\varepsilon > 0$. For the square annulus it can be shown that $\delta \ge \frac{9}{64}$ for any $\varepsilon > 0$, unlike the case of the usual annulus where $\delta \to 0$ as $\varepsilon \to 0$. On the other hand, the *randomly oriented* square annulus does have small δ by Lemma 1, so in this case we do obtain percolation for any fixed $\eta > 0$ if ε is sufficiently small.

2 Lower bound

Before we give the proof of Theorem 2, we first note that one can give a positive lower bound on η that is necessary for an infinite directed path to exist in $\mathcal{G}$. Recall that all regions A are assumed to lie within a ball of radius r_0 about 0, and the average number of neighbours is $1 + \eta$.

Theorem 5. *Suppose that $\eta < (2K)^{-K}/3$ where $K = \lceil (4r_0)^d + 1 \rceil$. Then there is almost surely no infinite directed path in $\mathcal{G}$.*

Proof. Fix x_0 in the process. Let N_i be the number of points at graph distance i from x_0 in $\mathcal{G}$. We need to show that, almost surely, $N_i = 0$ for some i. For all i, $\mathbb{E}(N_{i+1}|N_i) \le (1+\eta)N_i$, since this is true even conditioning on the choice of $A(x)$ and the Poisson process in $A(x)$ for all x within graph distance $i - 1$ of x_0. (All the points at graph distance $i + 1$ lie outside these regions and are contained within a union of N_i regions, each of expected volume $1 + \eta$.) Thus $\mathbb{E}N_{i+1} = \mathbb{E}(\mathbb{E}(N_{i+1} \mid N_i)) \le (1+\eta)\mathbb{E}N_i$. In particular, when $\eta < 0$,

$\mathbb{E}N_i \to 0$ as $i \to \infty$, and so almost surely some $N_i = 0$. Hence we may assume $\eta \geq 0$. Now $\mathbb{E}(N_2 \mid N_1) \leq (1 + \eta)N_1$. All the N_2 points at graph distance two from x_0 lie within $2r_0$ of x_0 and outside $A(x_0)$. Thus we also have $\mathbb{E}(N_2 \mid N_1) \leq (4r_0)^d$, since this is true even conditioning on $A(x_0)$ and the Poisson process inside $A(x_0)$. Combining, we have

$$\begin{aligned} \mathbb{E}N_2 = \mathbb{E}(\mathbb{E}(N_2 \mid N_1)) &\leq \mathbb{E}(\min\{(1 + \eta)N_1, (4r_0)^d\}) \\ &\leq (1 + \eta)\mathbb{E}N_1 - 1.\mathbb{P}\left(N_1 \geq (4r_0)^d + 1\right) \\ &\leq (1 + \eta)^2 - \mathbb{P}\left(N_1 \geq (4r_0)^d + 1\right). \end{aligned}$$

Let $A = A(x_0)$ be the neighbourhood for x_0. Then, since $|A| < (2r_0)^d$ and $\mathbb{E}|A| \geq 1$,

$$\mathbb{P}(|A| > \tfrac{1}{2}) > \frac{1}{2(2r_0)^d}.$$

Also, conditional on $|A| > \frac{1}{2}$, the number of points in A stochastically dominates a Poisson distribution with mean $\frac{1}{2}$. Thus

$$\mathbb{P}(N_1 \geq (4r_0)^d + 1 \mid |A| > \tfrac{1}{2}) \geq \frac{e^{-1/2}}{2^K K!}$$

where $K = \lceil (4r_0)^d + 1 \rceil$. Therefore,

$$\mathbb{P}(N_1 \geq (4r_0)^d + 1) \geq \frac{1}{2(2r_0)^d} \cdot \frac{e^{-1/2}}{2^K K!} > (2K)^{-K} > 3\eta > 2\eta + \eta^2,$$

by assumption on η. Hence, $\mathbb{E}N_2 < (1 + \eta)^2 - (2\eta + \eta^2) = 1$. However, it is clear that $\mathbb{E}(N_{i+2} \mid N_i) \leq (\mathbb{E}N_2)N_i$, so $\mathbb{E}N_{2i} \leq (\mathbb{E}N_2)^i$ which tends to 0 as $i \to \infty$. The result now follows. $\qquad\square$

3   Proof of Theorem 2

First we convert the definition of δ into a more usable form.

Lemma 6. *We can define sets $B(x_i)$ depending on $A(x_i)$ so that if $z \notin B(x_i)$ and A is distributed according to $\mathcal{D}$ then, conditional on any value of $A(x_i)$ we have*

$$\mathbb{E}|(z + A) \cap A(x_i)| \leq \sqrt{\delta}\,\mathbb{E}|A|.$$

Moreover, for all $z \in \mathbb{R}^d$ and $\mathcal{D}$-almost all choices of A and $A(x_i)$,

$$|(z \pm A) \cap B(x_i)| \leq \sqrt{\delta}\,|A|.$$

Proof. Fix $A(x_i)$ and define

$$B(x_i) = \{z \in \mathbb{R}^d : \mathbb{E}|(z + A) \cap A(x_i)| > \sqrt{\delta}\,\mathbb{E}|A|\,\}.$$

The first inequality then holds automatically. Now let X be uniformly distributed in A and let Y be distributed according to the distribution $\mathcal{D}_c$. Then for either choice of $\pm$ sign (fixed throughout), and almost all choices of A and $A(x_i)$,

$$
\begin{aligned}
\delta &\geq \mathbb{P}(\pm X + Y \in A(x_i) + z) \\
&\geq \Big(\mathbb{P}(\pm X + Y \in A(x_i) + z \mid \pm X \in B(x_i) + z) \Big)\Big(\mathbb{P}(\pm X \in B(x_i) + z) \Big) \\
&= \Big(\mathbb{P}(W + Y \in A(x_i) \mid W \in B(x_i)) \Big)\Big(\mathbb{P}(\pm X \in B(x_i) + z) \Big) \\
&= \Big(\mathbb{E}(|(W + A') \cap A(x_i)| \mid W \in B(x_i))/\mathbb{E}|A'| \Big)\Big(| \pm A \cap (B(x_i) + z)|/|A| \Big) \\
&\geq \sqrt{\delta}\,|((-z) \pm A) \cap B(x_i)|/|A|
\end{aligned}
$$

where A' is distributed according to $\mathcal{D}$ and $W = \pm X - z$. Hence $|(z \pm A) \cap B(x_i)| \leq \sqrt{\delta}\,|A|$ for all $z \in \mathbb{R}^d$. $\qquad\square$

The sets $B(x_i)$ give regions around x_i that we want to avoid, since if $x_j \in B(x_i)$ then the neighbourhoods of x_j and x_i may have a large intersection. In this case we will typically have too few 'new' neighbours of x_j, say, that we have not already encountered when looking for neighbours of x_i. Ideally we would like to have $B(x_i) = \emptyset$, but in general all we can guarantee is that $B(x_i)$ has small intersection with every region $A(x_k)$.

We introduce the following parameters of the distribution $\mathcal{D}_0$ governing N, the number of neighbours of a point.

$$\mathbb{E}N = 1 + \eta, \qquad \mathrm{Var}N = \sigma^2, \qquad \mathbb{P}(N = 0) = p_0.$$

Since the regions A are bounded, N is stochastically bounded by a Poisson variable with finite mean. Hence all moments, including $\mathrm{Var}N$, are finite, and $p_0 > 0$. By Theorem 5, if $\eta \leq 0$ then there will almost surely be no infinite directed path in $\mathcal{G}$, hence we shall always assume $\eta > 0$. We also define $\lambda > 0$ so that

$$\mathbb{E}e^{-\lambda N} = e^{-\lambda}. \tag{6}$$

Note that for large x, $\mathbb{E}e^{-xN}$ tends to $p_0 > 0$, and for small x, $\mathbb{E}e^{-xN} = 1 - x\mathbb{E}N + O(x^2) = 1 - x(1 + \eta) + O(x^2) < e^{-x}$. Hence by continuity, λ does indeed exist.

Lemma 7. *If A is distributed according to $\mathcal{D}$ and $\eta > 0$ then*

$$\eta = \mathbb{E}|A| - 1, \qquad \sigma^2 = \mathrm{Var}|A| + \mathbb{E}|A|, \qquad p_0 \geq e^{-(1+\eta)}, \qquad \lambda \geq \frac{2\eta}{\sigma^2 + (1 + \eta)^2}.$$

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Proof. Let A be distributed according to $\mathcal{D}$, so that N, conditioned on A, is distributed as a Poisson variable with mean $|A|$. For the first equation $1 + \eta = \mathbb{E}N = \mathbb{E}(\mathbb{E}(N \mid A)) = \mathbb{E}|A|$. Also, $\sigma^2 + (\mathbb{E}N)^2 = \mathbb{E}(N^2) = \mathbb{E}(\mathbb{E}(N^2 \mid A)) = \mathbb{E}(|A| + |A|^2) = \text{Var}|A| + (\mathbb{E}|A|)^2 + \mathbb{E}|A|$. Since $\mathbb{E}N = \mathbb{E}|A|$ we get the second equality. The probability that $N = 0$ is $\mathbb{E}(\mathbb{P}(N = 0 \mid A)) = \mathbb{E}e^{-|A|} \geq e^{-(1+\eta)}$ by convexity of e^{-x}. Finally, N is non-negative and $e^{-x} \leq 1 - x + x^2/2$ for $x \geq 0$. Hence $\mathbb{E}e^{-\lambda N} \leq 1 - \lambda\mathbb{E}N + \lambda^2\mathbb{E}(N^2)/2$. If $0 < \lambda < 2\eta/\mathbb{E}(N^2)$ then $\mathbb{E}e^{-\lambda N} < 1 - \lambda(1 + \eta) + \lambda\eta = 1 - \lambda < e^{-\lambda}$, contradicting the definition of λ. Thus $\lambda \geq 2\eta/\mathbb{E}(N^2) = \frac{2\eta}{\sigma^2+(1+\eta)^2}$. $\qquad\square$

Our aim will be to compare the percolation process with an oriented bond percolation on $\mathbb{Z}^2$. First we shall use a linear transformation to make $\mathcal{D}$ somewhat more symmetric. If we apply a volume-preserving linear transformation to $\mathbb{R}^d$ then the Poisson process will again give us a Poisson process of intensity 1. The expression (2) defining δ is unchanged under any such transformation, as are the parameters η, σ, λ, and p_0, which depend only on $\mathcal{D}_0$. If Y is distributed according to $\mathcal{D}_c$, then $\mathbb{E}((Y \cdot u)^2)$ is a positive definite quadratic form in u, so by a suitable volume-preserving linear transformation we can ensure that $\mathbb{E}((Y \cdot u)^2) = r_m'^2\|u\|^2$ for all u and some r_m', the analogue of r_m for the transformed process. All the parameters of Theorem 2 remain unaltered under this transformation except for r_0 and r_m. Before this transformation, $\mathbb{E}((Y \cdot u)^2) \geq r_m^2\|u\|^2$ and $|Y \cdot u| \leq r_0\|u\|$ for all u, so

$$r_0/r_m \geq \sup_{u\neq0}\operatorname{ess\,sup}_Y |Y \cdot u|/\sqrt{\mathbb{E}((Y \cdot u)^2)} = \sup_L \operatorname{ess\,sup}_Y |L(Y)|/\sqrt{\mathbb{E}(L(Y)^2)},$$

where L runs over all non-zero linear functionals. This last expression however is invariant under any linear transformation, and is equal to r_0'/r_m' where r_0' is the r_0 for the transformed process. Hence $r_0'/r_m' \leq r_0/r_m$. Since Theorem 2 depends on r_m and r_0 only via this ratio, it is enough to prove the result for this transformed version.

If the distribution $\mathcal{D}_c$ has drift we shall however apply a second volume-preserving linear transformation. We shall define a large constant $C = C(\sigma, \eta) > 1$ which is a function of σ and η only (and hence is unaffected by any volume-preserving transformation on $\mathbb{R}^d$). Now $\text{Var}(Y \cdot u) + (Cr_0/r_m)^2(\mathbb{E}Y \cdot u)^2$ is also a positive definite quadratic form in u, so we can apply a volume-preserving transformation so that

$$\text{Var}(Y \cdot u) + (Cr_0/r_m)^2(\mathbb{E}Y \cdot u)^2 = \mathbb{E}((Y \cdot u)^2) + ((Cr_0/r_m)^2 - 1)(\mathbb{E}Y \cdot u)^2 = r_s^2\|u\|^2$$

for some constant $r_s > 0$. Since $Cr_0/r_m > 1$, this transformation compresses vectors in the direction of $\mathbb{E}Y$ and expands vectors orthogonal to $\mathbb{E}Y$ by a factor r_s/r_m. Hence at worst r_0 increases by a factor r_s/r_m. From now on we shall replace r_0 by $r_1 = r_0r_s/r_m$ and r_m by r_s in the statement of Theorem 2, noting that r_1 is still a bound on the size of the transformed regions A_{x_i}. Finally, we shall rotate the process so that the drift $\mathbb{E}Y$, if non-zero, points in the direction $(1, 1, 0, 0, \dots)$ in $\mathbb{R}^d$. Define $R = Cr_1$. Then to summarise, in our transformed process

$$r_s^2\text{Var}(Y \cdot u) + R^2(\mathbb{E}Y \cdot u)^2 = r_s^4 \quad \text{and} \quad |Y \cdot u| \leq r_1, \tag{7}$$

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for any unit vector u, and we need to show that $\mathcal{G}$ almost surely has an infinite directed path under the assumption

$$\delta < c r_s^9 r_1^{-9} \eta^{16} \sigma^{-32}. \tag{8}$$

Partition $\mathbb{R}^2$ into $6R \times 6R$ squares, and let the site $x \in \mathbb{Z}^2$ correspond to the cylinder $C_x = (6Rx + [-3R, 3R)^2) \times \mathbb{R}^{d-2}$ in $\mathbb{R}^d = \mathbb{R}^2 \times \mathbb{R}^{d-2}$. More generally, write $C_{x,k}$, $k = 1, 2, 3$, for the cylinder $C_{x,k} = (6Rx + [-kR, kR)^2) \times \mathbb{R}^{d-2}$ (see Figure 1 for the case $d = 2$). Our bonds xy in $\mathbb{Z}^2$ will correspond to certain good events in the corresponding cylinder $C_x \cup C_y$ with 2-dimensional $6R \times 12R$ rectangular cross section. Throughout most of what follows we shall mostly be only interested in the first two coordinates of the points of our process, except when intersecting regions $A(x_i)$, in which case we use all $d \geq 2$ dimensions.

Roughly speaking, these good events will be the ability to get from some given set P of n points near the middle of C_x to every point of a set P' of size n near the middle of C_y by paths that lie entirely within $C_x \cup C_y$. Here n is some fixed large number that will be determined latter. Using n points instead of just one increases the chances of success since in practice it is very unlikely that there is a path from a given single point in C_x to C_y. But if there is, then there are often many points in C_y that we can reach. Nevertheless, provided we choose our sets consistently (so the P' for x becomes the P for y), an infinite directed path in $\mathbb{Z}^2$ from x will result in an infinite path in $\mathcal{G}$ from at least one of the points of P.

To construct these paths we will need to 'explore' the graph $\mathcal{G}$. When we encounter a vertex z there is a random choice of $A(z)$ and Poisson process within $A(z)$. If we condition on these we effectively fix these choices. When we encounter subsequent vertices w, the choice of $A(w)$ is independent of this conditioning event, but the Poisson process in $A(w)$ is already determined in $A(z) \cap A(w)$. However, the Poisson process in $A(z) \setminus A(w)$ is independent of the conditioning event. Thus we can imagine 'growing' $\mathcal{G}$ by at each step fixing $A(w)$ and the Poisson process in the subset of points of $A(w)$ that do not lie in any previously seen $A(z)$. Since the previous $A(z)$ contain vertices of $\mathcal{G}$ that we have encountered already, we want most of $A(w)$ to be new. To do this we must control the number of regions $A(z)$ we have previously looked at (i.e., conditioned on). Hence when constructing our paths from P in $C_x \cup C_y$ we shall only allow ourselves to 'test' regions $A(z)$ around at most N points (N to be determined below). The set of points we test will be called Q'. Also, prior to constructing these paths there will be a set Q of up to $3N$ points in $C_x \cup C_y$ that have been tested when constructing earlier bonds, and we shall need to avoid the regions $A(z)$ about these points as well. Since we wish to avoid points in $A(z)$ for $z \notin C_x \cup C_y$ we shall only consider points at least r_1 from the boundary of this set, and so we write $C_{xy}^o = \{z \in C_x \cup C_y : d(z, \partial(C_x \cup C_y)) \geq r_1\}$. We shall assume that we can get to $x \in \mathbb{Z}^2$ from the origin in our percolation on $\mathbb{Z}^2$, and hence we can get to every point in P. In doing so we have fixed points of $P \cup Q$, the choice of $A(z)$ for $z \in Q$, and the state of the Poisson process in $A(z)$ for $z \in Q$. Note in particular
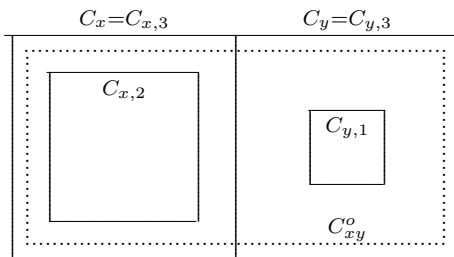
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Figure 1: Cylinder sets C_x and C_y.

that we have conditioned on an event that implies that there are at least n points in C_x. Having conditioned on this data we still have a Poisson process in the remaining region of C_{xy}^o and a choice of $A(z)$ for points in this process as well as for $z \in P$. Points outside of C_{xy}^o will be ignored when considering the bond xy of $\mathbb{Z}^2$.

Now we make things more formal. Fix points $P = \{x_1, \ldots, x_n\}$ and $Q = \{y_1, \ldots, y_k\}$, $k \le 3N$, in $C_x \cup C_y$ corresponding to an bond xy of $\mathbb{Z}^2$, where $y = x + (0,1)$ or $x + (1,0)$. Fix the choice of neighbourhood regions $A(y_j)$, $y_i \in Q$, and the Poisson process within these $A(y_j)$. In what follows, everything will be conditioned on this data. Some regions outside of C_{xy}^o will also be conditioned on due to the construction of previous bonds. Since we shall not consider points outside of C_{xy}^o when considering the bond xy, it will be convenient to assume that we have conditioned on the entire process outside of C_{xy}^o as well. We shall not however condition on the choice of regions $A(x_i)$, $x_i \in P$, or on the Poisson process in the rest of C_{xy}^o. We shall make the following assumptions.

a) The points x_i lie in the central cylinder $C_{x,2}$ for all $x_i \in P$,

b) $x_i \notin B(y_j)$ for all $x_i \in P$, $y_j \in Q$, (9)

c) $\mathbb{P}(x_i \in B(x_j)) \le \sqrt[4]{\delta}$ for all $x_i, x_j \in P$, $i \ne j$.

In c) the choice of A_{x_j} and hence $B(x_j)$ is random, and the probability refers to this choice which is made in accordance with the distribution $\mathcal{D}$. Assumptions b) and c) will be required to ensure that the neighbourhoods $A(z)$ that we construct are unlikely to overlap too much. Assumption a) avoids problems that may occur near the boundary of C_{xy}^o.

Given the situation described above, we shall construct sets of points $P' = \{x_1', \ldots, x_r'\}$ and $Q' = \{y_1', \ldots, y_{k'}'\}$, $r + k' \le N$, of the Poisson process in C_{xy}^o with the following properties.

a') The points x_i' lie in the central cylinder $C_{y,2}$ for all $x_i' \in P'$.

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b′) $x_i' \notin B(z)$ for all $x_i' \in P'$, $z \in P \cup Q \cup Q'$.

c′) $\mathbb{P}(x_i' \in B(x_j')) \leq \sqrt[4]{\delta}$ for all $x_i', x_j' \in P'$, $i \neq j$.

d′) $z \notin A(y_i)$ for all $z \in P' \cup Q'$, $y_i \in Q$.

e′) For all $x_i' \in P'$, some $x_j \in P$ is joined to x_i' by a sequence of points $y_{i_1}', \ldots, y_{i_t}'$ of Q'.

The construction will depend on the choice of $A(z)$ and the Poisson process restricted to $A(z)$ for $z \in P \cup Q'$, but will not depend on the values of $A(z)$ for $z \in P'$. Hence the probability in c′), which is over all choices of $A_{x_j'}$ according to the distribution $\mathcal{D}$, makes sense.

We shall declare the bond $\vec{xy}$ *open with respect to P and Q* if in this construction we can take $r = |P'| = n$. In this case conditions a′)–c′) ensure that P' and Q' can be used in the construction of the next bond $\vec{yz}$ of $\mathbb{Z}^2$. (The Q for $\vec{yz}$ will include any points of $P \cup Q \cup Q'$ that lie in C_{yz}.) Note that the openness of $\vec{xy}$ depends on the choices of P and Q as well as the restriction of the process to the region C_{xy}^o. Condition d′) ensures that we never look at the process in the region $\bigcup_{y_i \in Q} A(y_i)$, as these regions will have been tested earlier. Condition e′) ensures that every point of P' is reachable from some point of P in $\mathcal{G}$, so that an infinite path in the $\mathbb{Z}^2$ process will ensure an infinite path in $\mathcal{G}$.

The sets P and Q will depend on the construction of previous bonds, which will introduce complex dependencies between the bonds. Nevertheless, we start by showing that an individual bond is open with high probability, regardless of the choice of P and Q and regardless of the Poisson process in $\bigcup_{y_i \in Q} A(y_i)$ or outside of C_{xy}^o.

The proof of the bound is complicated by the fact that the regions $A(z)$ intersect, so we shall first consider the simpler case when we ignore these intersections and model the percolation by a branching process (see for example [1]). We shall generally refer to the points of a branching process as *nodes* to avoid confusion with the points of our Poisson process. We shall first prove two simple results about branching processes. The parameters η, σ, p_0, and λ are as defined before Lemma 7.

Lemma 8. *Consider a branching process where at each step each node branches into several new nodes independently according to the distribution $\mathcal{D}_0$. Let N_t be the number of nodes at time $t > 0$. Then $\mathbb{P}(N_t \geq (1+\eta)^t) \geq \eta p_0^2 / \sigma^2$.*

Proof. It is easy to show by induction on t that

$$\mathbb{E}N_t = (1+\eta)^t \qquad \text{and} \qquad \text{Var}N_t = \sigma^2 (1+\eta)^{t-1}((1+\eta)^t - 1)/\eta.$$

Let $X_t = N_t/\mathbb{E}N_t$, so that $\mathbb{E}X_t = 1$ and $\text{Var}X_t < \sigma^2/\eta$. By Cauchy-Schwarz,

$$\mathbb{E}(I_{X_t \geq 1})\mathbb{E}((X_t - 1)^2 I_{X_t \geq 1}) \geq (\mathbb{E}((X_t - 1)I_{X_t \geq 1}))^2$$

Now
$$\mathbb{E}((X_t - 1)^2 I_{X_t \geq 1}) \leq \mathbb{E}((X_t - 1)^2) = \mathrm{Var}X_t < \sigma^2/\eta,$$
$$\mathbb{E}((X_t - 1)I_{X_t \geq 1}) = \mathbb{E}((1 - X_t)I_{X_t \leq 1}) \geq \mathbb{P}(X_t = 0) \geq \mathbb{P}(N_1 = 0) = p_0.$$

Hence $\mathbb{P}(N_t \geq (1 + \eta)^t) = \mathbb{E}(I_{X_t \geq 1}) \geq \eta p_0^2/\sigma^2.$ $\qquad \square$

Lemma 9. *Consider a branching process where at each step each node branches into several new nodes independently according to the distribution $\mathcal{D}_0$. Suppose also that we randomly remove nodes, if necessary, so there are at most K nodes at each step and assume $T \leq \frac{\lambda}{2}e^{\lambda(K-1)}$. Then the probability that there is at least one node at time T is at least $1 - e^{-\lambda/2}$.*

Proof. Let N_t be the number of nodes at time t and consider the random variable X_t defined by $X_t = e^{-\lambda N_t}$. Now

$$N_{t+1} = \min(N'_{t+1}, K), \qquad \text{where} \quad N'_{t+1} = \sum_{i=1}^{N_t} Y_i$$

and Y_i are independent random variables with distribution $\mathcal{D}_0$. Recall that λ is defined so that
$$\mathbb{E}e^{-\lambda Y_i} = e^{-\lambda}.$$

Hence
$$\mathbb{E}(e^{-\lambda N'_{t+1}} \mid N_t) = \prod_{i=1}^{N_t}\mathbb{E}e^{-\lambda Y_i} = \prod_{i=1}^{N_t}e^{-\lambda} = e^{-\lambda N_t} = X_t$$

But if $N_{t+1} \neq N'_{t+1}$ then $K = N_{t+1} < N'_{t+1}$. Therefore

$$0 \leq \mathbb{E}(e^{-\lambda N_{t+1}} - e^{-\lambda N'_{t+1}} \mid N_t) \leq e^{-\lambda K}$$

so $\mathbb{E}(X_{t+1} \mid N_t) \leq X_t + e^{-\lambda K}$, and thus $\mathbb{E}X_{t+1} \leq \mathbb{E}X_t + e^{-\lambda K}$. But $\mathbb{E}X_t \geq \mathbb{P}(N_t = 0)$ and $\mathbb{E}X_0 = e^{-\lambda N_0} = e^{-\lambda}$. Hence

$$\mathbb{P}(N_T = 0) \leq \mathbb{E}X_T \leq e^{-\lambda} + Te^{-\lambda K} \leq (1 + \tfrac{\lambda}{2})e^{-\lambda} \leq e^{-\lambda/2}$$

and $\mathbb{P}(N_T > 0) \geq 1 - e^{-\lambda/2}$ as required. $\qquad \square$

We now consider a simplified version of our percolation process in which each step is independent of all previous steps. We define a branching process of nodes. Each node is assigned a region A_v independently of all previous nodes according to the distribution $\mathcal{D}$, and then branches into a number of new nodes, where the number of child nodes is Poisson distributed with mean $|A_v|$. As a consequence, the number of child nodes has overall distribution $\mathcal{D}_0$. Also, for each child node u of v we choose δ_u uniformly from the set A_v independently of all other δ_u's. Unconditioned on A_v, δ_u then has probability distribution $\mathcal{D}_c$. Fix a position z_0 in $\mathbb{R}^d$ for the root node and define the position z_v of a node v to be $z_0 + \sum_u \delta_u$ where the sum runs over all predecessors u of v back to the root node. Let $\mathcal{T}$ be the random graph with vertices z_v and edges $\vec{z_v z_u}$ for all child

nodes u of v. Set $\mathcal{T}^t$ to be the set of nodes that are t steps from the root node in the graph $\mathcal{T}$. The process $\mathcal{T}$ approximates the percolation process $\mathcal{G}$, but it differs in that the distribution of points (child nodes) in $A(z_v) = A_v + z_v$ is independent of the process up to that point, whereas in $\mathcal{G}$ the points in $A(z_v)$ will depend on points in previously encountered regions $A(z_u)$ where they intersect. Effectively, the difference between $\mathcal{T}$ and $\mathcal{G}$ is that when we encounter a node z_u in $\mathcal{T}$ we 'regenerate' the Poisson process in the whole of $A(z_u)$, whereas in $\mathcal{G}$ we only generate a Poisson process in the region of $A(z_u)$ that we have not already seen. To simplify the notation, we shall generally identify a node u with its position $z_u \in \mathbb{R}^d$, so for example we shall denote $A(z_u)$ by $A(u)$.

In $\mathcal{T}$, any path from the root gives rise to a random walk in $\mathbb{R}^d$. We shall analyse this walk by comparison with a Brownian motion. Recall that a Brownian motion (with drift) is a continuous time stochastic process $B_t \in \mathbb{R}$ such that for all $t_1 > t_2$, $B_{t_1} - B_{t_2}$ is given by a normal distribution with mean $\beta(t_2 - t_1)$ and variance $\gamma(t_2 - t_1)$ and is independent of B_t for $t < t_1$. We call β the *drift* and γ the *unit time variance* of B_t. We start with a well known result.

Lemma 10. *If B_t is a a Brownian motion with drift β and unit time variance γ, then the probability of B_t hitting 1 before hitting 0 starting at $x \in [0,1]$ is given by*

$$f(x) = \frac{1 - e^{-2x\beta/\gamma}}{1 - e^{-2\beta/\gamma}} \quad (\beta \neq 0), \qquad or \qquad f(x) = x \quad (\beta = 0).$$

Proof. Let $f(x)$ be the solution of the equation

$$\frac{\gamma}{2}\frac{d^2 f}{dx^2} + \beta\frac{df}{dx} = 0, \qquad f(0) = 0, \quad f(1) = 1.$$

Then $\mathbb{E}(f(B_{t+\delta t}) - f(B_t) \mid B_t = x) = \mathbb{E}(f(x + Z) - f(x))$ where Z is normal with mean $\beta\,\delta t$ and variance $\gamma\,\delta t$. By solving the above differential equation, we see that $|f'''(x)| = O(C^{|x|})$ for some C. Thus $f(x + Z) - f(x) = Zf'(x) + \frac{1}{2}Z^2 f''(x) + O(Z^3 C^{|Z|})$, and

$$\mathbb{E}(f(x + Z) - f(x)) = \beta\,\delta t f'(x) + \tfrac{1}{2}(\gamma\,\delta t + \beta^2 \delta t^2)f''(x) + O(\delta t^{3/2}) = o(\delta t).$$

Taking $\delta t \to 0$ and summing between $t = t_1$ and $t = t_2$, we see that $\mathbb{E}(f(B_{t_2}) \mid f(B_{t_1})) = f(B_{t_1})$ for all $t_2 > t_1$, i.e., $f(B_t)$ is a Martingale. Since f is bounded on $[0,1]$ and B_t is almost surely continuous, $f(x) = \mathbb{E}f(B_0) = \mathbb{E}f(B_T)$ where T is the stopping time $\inf\{t : B_t = 0 \text{ or } 1\}$. But $f(0) = 0$ and $f(1) = 1$, so this is just the probability that B_t hits 1 before hitting 0. Solving the above differential equation gives the result. $\qquad\square$

Lemma 11. *There exist absolute constants $c_1, c_2, c_3 > 0$ with the following property. Assume we are given a random walk Z_i in $\mathbb{R}^2$ with i.i.d. steps $Y_i = Z_{i+1} - Z_i$. Assume the mean step size $\mathbb{E}Y_i$ is either zero or is in the direction $(1,1)$ and there is a constant $r > 0$ such that for any unit vector u, $r^2\mathrm{Var}(Y_i \cdot u) + (\mathbb{E}Y_i \cdot u)^2 = r^4$. Assume further that*
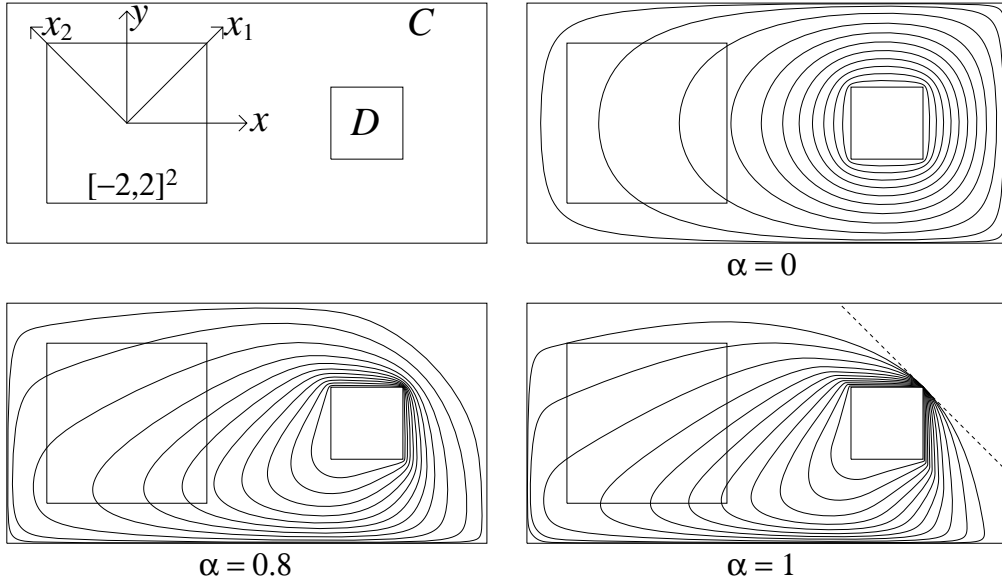
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Figure 2: Function f_α in C. Curves are the contours $f_\alpha = 0.005, 0.03, 0.1, 0.2, \ldots, 0.9$. Values of f_α are strictly positive in C except when $\alpha = 1$ and $x_1 \geq 7.8/\sqrt{2}$.

the maximum step size is at most c_1, and the random walk starts at $Z_0 \in [-2, 2] \times [-2, 2]$. Then the probability of the random walk entering $[5, 7] \times [-1, 1]$ before getting within c_1 of the boundary of the rectangle $[-3, 9] \times [-3, 3]$ and before time c_2/r^2 is at least c_3.

Proof. Introduce coordinates $x_1 = (x + y)/\sqrt{2}$, $x_2 = (y - x)/\sqrt{2}$, so the (x_1, x_2) co-ordinates are obtained from the (x, y) coordinates by a rotation of $45°$. Then $\mathbb{E}Y_i$ lies on the x_1-axis. Assume $\|\mathbb{E}Y_i\| = \alpha r^2$, so the variance of the step size is $(1 - \alpha^2)r^2$ in the x_1-direction and r^2 in the x_2-direction. Note that the maximum step size must be at least r, so $r \leq c_1$. Consider a 2-dimensional Brownian motion with independent x_1 and x_2 coordinates, drift α in the x_1-direction, unit time variance $1 - \alpha^2$ in the x_1-direction, and unit time variance 1 in the x_2-direction. Let $f_\alpha(z)$ be the probability of this Brownian motion starting at $z \in \mathbb{R}^2$ hitting $D = [5.1, 6.9] \times [-0.9, 0.9]$ before leaving $C = [-3, 9] \times [-3, 3]$ (see Figure 2). Then arguing as in Lemma 10 one can show that $f_\alpha(z)$ satisfies the following PDE.

$$\tfrac{1}{2}\frac{\partial^2 f_\alpha}{\partial x_2^2} + \tfrac{1}{2}(1 - \alpha^2)\frac{\partial^2 f_\alpha}{\partial x_1^2} + \alpha \frac{\partial f_\alpha}{\partial x_1} = 0.$$

Now for $\alpha \in [0, 1]$, $f_\alpha(z)$ is a smooth function in the interior of $C \setminus D$ which is 1 on ∂D, 0 on ∂C, and $0 \leq f_\alpha(z) \leq 1$ on $C \setminus D$. Note that $f_\alpha(z) > 0$ in $[-2, 2] \times [-2, 2]$ for all α, indeed $f_\alpha(z) > 0$ for all z in the interior of $C \setminus D$ except in the region $x_1 \geq 7.8/\sqrt{2}$ when

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$\alpha = 1$. Choose $c_3 > 0$ so that

$$3c_3 \leq \inf\{f_\alpha(x, y) : (x, y) \in [-2, 2] \times [-2, 2], \, \alpha \in [0, 1]\}. \tag{10}$$

We shall assume $c_3 < 0.1$. As z approaches ∂C, $f_\alpha(z)$ tends to zero, uniformly in both z and α. To see this, note that, for $y \in [-3, -2]$, $f_\alpha(x, y)$ is at most the probability of the Brownian motion hitting $y = -2$ before $y = -3$, since if it hits $y = -3$ then it has definitely hit ∂C, whereas to hit ∂D it must have hit $y = -2$ first. If we project the Brownian motion onto the y-coordinate, we obtain a Brownian motion with drift $\alpha/\sqrt{2}$ and unit time variance $1 - \alpha^2/2$, so by Lemma 10 we have $f_\alpha \leq (1 - e^{-\varepsilon\alpha\sqrt{2}/(2-\alpha^2)})/(1 - e^{-\alpha\sqrt{2}/(2-\alpha^2)})$ when we are within ε of the $y = -3$ side of ∂C. This expression is maximised when $\alpha = 1$, so we get $f_\alpha \leq (1 - e^{-\varepsilon\sqrt{2}})/(1 - e^{-\sqrt{2}}) \leq 1.87\varepsilon$. A similar argument applies starting with $y \in [2, 3]$, except in this case the drift helps us and the maximum occurs at $\alpha = 0$ with $f_\alpha \leq \varepsilon$ when we are within ε of $y = 3$. Projecting onto the x-coordinate deals with the remaining two sides of ∂C and gives smaller bounds. Thus if $c_4 > 0$ is sufficiently small ($c_4 \leq c_3/1.87$), we can assume that $|f_\alpha(x)| < c_3$ when $d(x, \partial C) < c_4$. We shall assume $c_4 < 0.1$. Since all derivatives of $f_\alpha(x)$ exist and are continuous on the interior of $C \setminus D$, we can assume the second and third directional derivatives $D_u^2 f_\alpha(x)$ and $D_u^3 f_\alpha(x)$ are bounded by a constant M, uniformly in u, x, and α, provided x is at distance at least $c_4/2$ from the boundary of $C \setminus D$. Provided Z_i is further than c_4 from $\partial C \cup \partial D$, we can estimate $\mathbb{E}(f_\alpha(Z_{i+1}) \mid Z_i)$ by approximating $f(x)$ near $x = Z_i$. Indeed, if $c_1 < c_4/2$ then Z_{i+1} must be at least $c_4/2$ from $\partial C \cup \partial D$ and so

$$f_\alpha(Z_{i+1}) - f_\alpha(Z_i) = \sum_{j=1}^{2} u_j \frac{\partial f_\alpha}{\partial x_j} + \frac{1}{2} \sum_{j,k=1}^{2} u_j u_k \frac{\partial^2 f_\alpha}{\partial x_j \partial x_k} + \theta(Y_i)\|Y_i\|^3$$

where $Y_i = (u_1, u_2)$ and $|\theta(Y_i)| \leq M/6$. However, $\mathbb{E}u_1 = \alpha r^2$, $\mathbb{E}u_2 = \mathbb{E}u_1 u_2 = 0$, $\mathbb{E}u_1^2 = (1 - \alpha^2)r^2 + \alpha^2 r^4$, and $\mathbb{E}u_2^2 = r^2$, so

$$\mathbb{E}(f_\alpha(Z_{i+1}) - f_\alpha(Z_i) \mid Z_i)$$
$$= \alpha r^2 \frac{\partial f_\alpha}{\partial x_2} + \frac{1}{2}((1 - \alpha^2)r^2 + \alpha^2 r^4)\frac{\partial^2 f_\alpha}{\partial x_2^2} + \frac{1}{2}r^2 \frac{\partial^2 f_\alpha}{\partial x_1^2} + \mathbb{E}(\theta(Y_i)\|Y_i\|^3)$$
$$= \frac{1}{2}\alpha^2 r^4 \frac{\partial^2 f_\alpha}{\partial x_2^2} + \mathbb{E}(\theta(Y_i)\|Y_i\|^3).$$

Since $r \leq c_1 < 1$ and $\alpha \leq 1$ we get

$$|\mathbb{E}(f_\alpha(Z_{i+1}) - f_\alpha(Z_i) \mid Z_i)| \leq \frac{1}{2}\alpha^2 r^4 M + \frac{1}{6}Mc_1 r^2 \leq Mc_1 r^2.$$

Run the random walk until the stopping time $T = \min\{i : d(Z_i, \partial C \cup \partial D) < c_4\}$. We shall now show that T is unlikely to be very large. Writing $(Z_t)_2$ for the x_2-coordinate of Z_t we have $\mathbb{E}((Z_{t+1})_2^2 \mid Z_t) = (Z_t)_2^2 + r^2$, so if we set $g(t) = \mathbb{E}((Z_{\min(t,T)})_2^2)$, then $g(t+1) = g(t) + r^2\mathbb{P}(T > t)$. Since $(Z_{\min(t,T)})_2^2$ is bounded above by $\sup_C x_2^2 = (12/\sqrt{2})^2 = 72$ and

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$\mathbb{P}(T > t)$ is monotonically decreasing in t, we have $tr^2 \mathbb{P}(T > t) \leq g(t+1) - g(1) \leq 72$ for all t. Hence if $c_2 \geq 72/c_3$, $\mathbb{P}(T > c_2/r^2) \leq c_3$. Since $f_\alpha(z)$ is bounded between 0 and 1,

$$
\begin{aligned}
\mathbb{E}(f_\alpha(Z_T)) &\geq \mathbb{P}(T \leq c_2/r^2) \mathbb{E}(f_\alpha(Z_{\min\{T, c_2/r^2\}})) \\
&\geq (1 - c_3)(f_\alpha(Z_0) - M c_1 r^2 (c_2/r^2)) \\
&\geq 0.9(3c_3 - M c_1 c_2),
\end{aligned}
$$

which is at least $2c_3$ for sufficiently small c_1 ($c_1 < 0.7c_3/M$ is enough). Hence if $p = \mathbb{P}(d(Z_T, \partial D) < c_4)$, then $(1)(p) + (c_3)(1-p) \geq \mathbb{E}(f_\alpha(Z_T)) \geq 2c_3$, so $p \geq c_3$. Therefore, with probability at least c_3, the random walk gets closer than c_4 to D (and hence enters $[5, 7] \times [-1, 1]$) before time c_2/r^2 and before getting within $c_4 > c_1$ of the boundary of the rectangle $[-3, 9] \times [-3, 3]$. $\qquad\square$

The numerical values of c_1, c_2, c_3 given by this proof are not very good. Indeed, simulations suggest that the bound on $3c_3$ given by (10) is just over 0.005, so we can take $c_3 = \frac{1}{600}$, $c_4 = \frac{1}{1200}$ and $c_2 = 43200$. One can show that for this c_4 we can set $M = 10^{23}$ (worst case is near the $(6.9, 0.9)$ corner of D when $\alpha = 1$), and then we can take $c_1 = 10^{-31}$.

Lemma 12. *Assume c_1, c_2, c_3 are as in Lemma 11. Consider the branching process $\mathcal{T}$ defined above where z_0 lies in $C_{x,2}$. Run $\mathcal{T}$ for time $T = \lfloor c_2 R^2/r_s^2 \rfloor$, except that at each step (if necessary) we remove nodes (randomly, independent of their positions) so there are at most K nodes left from $\mathcal{T}^t$. Assume $R \geq r_1/c_1$ and let $\mathcal{E}$ be the event that for a node w uniformly randomly chosen from $\mathcal{T}^T$, some ancestor v of w lies in the region $C_{y,1}$ and all ancestors of v lie in C_{xy}^o. Then $\mathbb{P}(\mathcal{E} \mid \mathcal{T}^T \neq \emptyset) \geq c_3$.*

Proof. Conditioning on $\mathcal{T}^T \neq \emptyset$, pick a node w uniformly at random from $\mathcal{T}^T$. Then for any $t < T$, the unique ancestor of w in $\mathcal{T}^t$ is equally likely to be any element of $\mathcal{T}^t$. (The number of descendants of each $u \in \mathcal{T}^t$ is independent of the process up to u.) Hence the locations of the nodes on the path from the root node to w form a random walk with steps taken from the distribution $\mathcal{D}_c$. If we project onto the first two coordinates, translate so that the centre of C_x is at the origin, and scale by a factor of $1/R$, this random walk will start in $[-2, 2] \times [-2, 2]$, have maximum step size $r_1/R \leq c_1$, and for any unit vector u, $r^2 \mathrm{Var}(Y \cdot u) + (\mathbb{E}Y \cdot u)^2 = r^4$, where $r = r_s/R$. The drift (if any) is in the $(1, 1)$ direction. By Lemma 11, with probability at least c_3 the walk enters $[5, 7] \times [-1, 1]$ before getting within c_1 of the boundary of $[-3, 9] \times [-3, 3]$ and before time $c_2(R/r_s)^2$. Thus with probability at least c_3 the original walk enters $C_{y,1}$ before getting within $c_1 R \geq r_1$ of the boundary of $C_x \cup C_y$ and by time T. $\qquad\square$

Theorem 13. *There exists an absolute constant $c > 0$ such that if $\delta < cr_s^9 r_1^{-9} \eta^{16} \sigma^{-32}$ then one can choose N, n, and $R = Cr_1$ such that the probability of a bond being open with respect to any P and Q satisfying conditions a)–c) of (9), conditioned on the choices of $A(z)$, $z \in Q$, and the Poisson process outside of $C_{xy}^o \setminus \bigcup_{z \in Q} A(z)$, is at least 0.9.*

Proof. The strategy of the proof is to find some path from $x_i \in P$ to some point x_i'' in $C_{y,1}$ in the $\mathcal{T}$ process, and by coupling, show that with reasonable probability, a corresponding path will exist in the $\mathcal{G}$ process. We shall then couple the percolation for R/r_1 further steps to obtain at least n points x_j'. These will lie in $C_{y,2}$ as we cannot travel more than distance R in R/r_1 steps. Since the probability of success is fairly small, we apply this process to each $x_i \in P$ and show that it is very likely that we will succeed for at least one x_i.

Run n independent truncated branching processes for T steps starting at the points $x_1, \ldots, x_n$ as in Lemma 12, where

$$T = \left\lfloor \frac{c_2 R^2}{r_s^2} \right\rfloor, \tag{11}$$

$$K = \left\lceil \frac{1}{\lambda} \log \frac{2T}{\lambda} + 1 \right\rceil. \tag{12}$$

Let $\mathcal{T} = \bigcup_{i=1}^n \mathcal{T}_i$ be the union of these processes, so that $\mathcal{T}$ is a union of n trees, with the i'th tree $\mathcal{T}_i$ starting at x_i. We shall couple $\mathcal{T}$ with $\mathcal{G}$ one level at a time, and within each level $\mathcal{T}^t$, sequentially run through all nodes in the previous level, and couple all the children of this node, one at a time (i.e., after coupling one node, we continue with each of its siblings before processing any other node). We shall declare some nodes u to be *good*. For these nodes, the positions and choice of A_u will have been successfully coupled so that they are equal to the corresponding values in the $\mathcal{G}$ process. However, we shall also require them to avoid certain sets $B(z)$, and for their sets $B(u)$ to avoid certain other nodes. We shall not attempt to couple the descendants of a node u or the process inside $A(u)$ unless u is good. (Technically, we couple the process independently with no requirement that the existence of v, position of v, or region $A(v)$ defined in the $\mathcal{T}$ process matches with anything in the $\mathcal{G}$ process when v is a descendant of a bad node u.)

The set $Q(u)$ will be the points of Q together with the set of all good nodes that have had all their children coupled before u. Before coupling u we will have fixed the Poisson process in $\bigcup_{z \in Q(u)} A(z)$ and outside C_{xy}^o. Let $P(u)$ be the set of good nodes found that are not in $Q(u)$. These are nodes which have uncoupled children, and in particular includes p_u, the parent of u. For these nodes (other than p_u) we have not yet fixed the Poisson process in the $A(z)$ (see Figure 3).

We shall ensure that we do not consider more than N new points of $\mathcal{G}$ in C_{xy}^o, where

$$N = n(KT + \lfloor R/r_1 \rfloor). \tag{13}$$

Hence in particular $|Q(u) \cup P(u)| \le N + |Q| \le 4N$ for all u.

Step 1: We start by coupling the choices of A_{x_i} in $\mathcal{T}$ to match those in $\mathcal{G}$ for each $x_i \in P$. Recall that A_{x_i} determines $B(x_i)$. Define $\mathcal{E}_i$ to be the event that $x_j \in B(x_i)$ for some $x_j \ne x_i$. By condition c), the probability of $\mathcal{E}_i$ is bounded above by

$$p_1 = n\sqrt[4]{\delta}. \tag{14}$$

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If $\mathcal{E}_i$ occurs we declare x_i to be bad, and we do not couple the process in $A(x_i)$, or any descendant of x_i. Otherwise x_i is good. After Step 1, but before processing the first child u of the first x_i, we have $Q(u) = Q$, and $P(u) = P_g$ is the subset of good nodes in P.

Step 2: We process each node of $\mathcal{T}$ in turn as described above, but before coupling any children of p_u we check if $p_u \in C_{y,1}$. In this case we do not couple any children of p_u, and p_u will remain in $P(v)$ for all subsequent nodes v.

Step 3: Couple the Poisson process in $A(p_u)$ in $\mathcal{T}$ with the same region in $\mathcal{G}$ so that they are the same where it does not overlap previously coupled regions $A(z)$, or lies in one of the areas we have conditioned on (i.e., they should agree in $C_{xy}^o \setminus \bigcup_{z \in Q(u)} A(z)$).

Step 4: For each child u of p_u, we couple the position of u in $\mathbb{R}^d$ to the position of a point in $\mathcal{G}$ provided

$$u \in C_{xy}^o, \qquad u \notin A(z) \text{ for all } z \in Q(u), \qquad u \notin B(z) \text{ for all } z \in Q(u) \cup P(u).$$

Otherwise the node u will be bad and we shall not couple the process in the region $A(u)$, or any descendant of u. Note that (by Step 3) the first two conditions are essential if we are to couple the $\mathcal{T}$ and $\mathcal{G}$ processes so that the locations of the nodes agree with the points in $\mathcal{G}$. The last condition ($u \notin B(z)$) will be used to ensure that the children of u are unlikely to lie in some previously coupled set $A(z)$.

Step 5: Couple A_u with the corresponding choice in $\mathcal{G}$. This now defines $B(u)$. We require

$$z \notin B(u) \text{ for all } z \in P(u).$$

Otherwise we again call u bad and ignore it and all its descendants. The reason is that if $z \in B(u)$ then coupling the descendants of z would be prejudiced.

Repeat Steps 4 and 5 for each child of p_u before continuing with the next parent node (with Step 2). Note that for $x_i \in P_g$ we do not need to (and cannot) insist that $x_i \notin A(z)$ for all $z \in Q$. However, by Step 1 and condition b), $x_i \notin B(z)$ for all $z \in Q \cup P_g$.

Assume $\mathcal{T}_i^T \neq \emptyset$ and pick a node uniformly at random from $\mathcal{T}_i^T$. We wish to estimate the probability that all nodes on the path from x_i to this node are good. Consider a node u on this path and assume all its predecessors are good. The parent p_u does not lie in $B(z)$ for any $z \in Q(u)$, since $Q(u) \subseteq Q(p_u) \cup P(p_u)$ and p_u is good (or since $Q(u) \subseteq Q \cup P_g$ and p_u is good in the case when $p_u = x_i \in P$). However $u - p_u$ is distributed according to $\mathcal{D}_c$, so by Lemma 6 and the definition of $\mathcal{D}_c$, the probability that $u \in A(z)$ for some $z \in Q(u)$ is bounded above by $|Q(u)|\sqrt{\delta} \leq 4N\sqrt{\delta}$.

By Lemma 6, the probability that $u \in B(z)$ for some $z \in Q(u) \cup P(u)$ is also at most $4N\sqrt{\delta}$. Note that if z is a sibling of u, then we must assume $A(p_u)$ is fixed, so we need the full strength of Lemma 6, that is that $|(A + p_u) \cap B(z)| \leq \sqrt{\delta}|A|$ for *almost all* A, not just averaged over A.
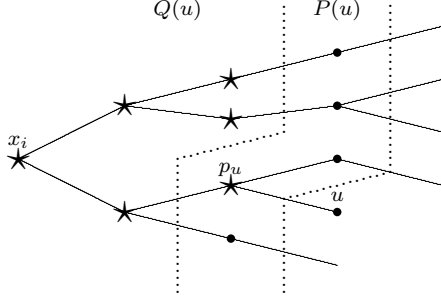
Figure 3: Situation just before processing node u of $\mathcal{T}$. All nodes $v \in P(u) \cup Q(u)$ have been coupled to points x_v in $\mathcal{G}$, and the choice of A_v has been matched to that of $A(x_v)$. For nodes marked with $\star$ we have also coupled the Poisson process in $A(x_v)$ where it does not overlap previously fixed regions.

Finally, the probability that $z \in B(u)$ for some $z \in P(u)$ is at most $4N\sqrt{\delta}$ since, even conditioned on A_u and A_{p_u}, the probability that a fixed z lies in $B(u)$ is the same as the probability that $z - u$ lies in $B_u = B(u) - u$, and $z - u$ is uniformly distributed in $(z - p_u) - A_{p_u}$. By Lemma 6 the probability that $z \in B(u)$ is then at most $\sqrt{\delta}$ for each z.

Hence, ignoring the possibility that we have left C_{xy}^o or entered $C_{y,1}$, the probability that $\mathcal{E}_i$ did not occur, but we failed to couple the path in $\mathcal{T}_i$ with $\mathcal{G}$, is at most $12NT\sqrt{\delta}$. We shall require that the parameters n and R are chosen below so that

$$12NT\sqrt{\delta} \le c_3/2, \tag{15}$$

where c_3 is the constant given by Lemma 11.

Now, conditioning on $\mathcal{T}_i^T \ne \emptyset$, the above path in $\mathcal{T}_i$ hits $C_{y,1}$ before leaving C_{xy}^o with probability at least c_3. Hence with probability at least $c_3 - c_3/2 = c_3/2$, either $\mathcal{E}_i$ occurred, or the coupling above stopped at a good node $x_i'' \in C_{y,1}$. Now put all the other coupled nodes of $\mathcal{T}_i$ into $Q(u)$ and run another $\mathcal{T}$ process starting at the points x_i'', $i = 1, \ldots, n$, that are good and lie in $C_{y,1}$. We run this second $\mathcal{T}$ process for another $\lfloor R/r_1 \rfloor$ steps. We couple this $\mathcal{T}$ process with $\mathcal{G}$ as above. Of course, this time we do not stop coupling if we hit $C_{y,1}$, and we cannot leave C_{xy}^o (or even $C_{y,2}$) in $\lfloor R/r_1 \rfloor$ steps. One minor difference is that in each level we do not couple the $A(u)$'s, and hence do not insist on $z \notin B(u)$ for z and u in the current level, until all nodes from that level are processed. If there are at least n nodes left in this level we stop and set P' equal to n of these nodes. Otherwise we apply Step 5 to all good nodes of this level (checking both $u \notin B(z)$ and $z \notin B(u)$ conditions of Step 4 and Step 5) before starting the next level. The reason for this complication is that we do not want to couple the choice of A_u's for $u \in P'$.

Assuming x_i'' exists, we now estimate the probability that a node u of this new process is bad, conditioned on all its predecessors being good, and on any event involving the

existence of its descendants. As above this probability is at most $12N\sqrt{\delta}$, and we shall require

$$12N\sqrt{\delta} < \eta/2. \tag{16}$$

Thus, the good nodes stochastically dominate a branching process with one-step mean $1 + \eta/2$, variance at most $\sigma^2 + (1+\eta)^2 - (1+\eta/2)^2 \leq \sigma^2 + 2 \leq 3\sigma^2$ (as $\eta \leq 1$) and probability of zero children at least p_0. Hence by Lemma 8 we will have more than n good descendants of x_i'' within time $\lfloor R/r_1 \rfloor$ with probability at least $\eta p_0^2/(6\sigma^2)$ provided

$$n \leq (1 + \eta/2)^{\lfloor R/r_1 \rfloor}. \tag{17}$$

If at any level we have found n good points (without the condition that $z \notin B(u)$ for $z \in P(u)$), then we stop the process, and set P' to be these n points. Any other good points found will be placed into the set Q'. The probability that either $\mathcal{E}_i$ occurs, or we get n good descendants of x_i in $C_{y,2}$, is now at least

$$p_2 = (1 - e^{-\lambda/2})\Big(\frac{c_3}{2}\Big)\Big(\frac{\eta p_0^2}{6\sigma^2}\Big), \tag{18}$$

where $1 - e^{-\lambda/2}$ bounds the probability that $\mathcal{T}_i^T \neq \emptyset$ (by Lemma 9), $c_3/2$ bounds the probability that given this either $\mathcal{E}_i$ occurs or we obtain a good x_i'', and $\eta p_0^2/(6\sigma^2)$ bounds the probability that a good x_i'' has n good descendants at some level.

Since $\mathbb{P}(\mathcal{E}_i) \leq p_1$, we obtain n good descendants of x_i with probability at least $p_2 - p_1$, regardless of the success or otherwise at any other x_j. Hence we will fail to obtain n good descendants in $C_{y,2}$ of any x_i with probability at most $(1 - (p_2 - p_1))^n \leq \exp(-(p_2 - p_1)n)$. This clearly bounds the probability of failing to obtain n good descendants from all x_i combined. We shall require

$$p_2 \geq 3.05/n, \tag{19}$$

and

$$n^2 \sqrt[4]{\delta} < 0.05, \tag{20}$$

so that $p_1 = n\sqrt[4]{\delta} \leq 0.05/n$ and $\exp(-(p_2 - p_1)n) \leq \exp(-3) \leq 0.05$. The number of points considered is at most nKT in the first $\mathcal{T}$ process and at most $n\lfloor R/r_1 \rfloor$ in the second $\mathcal{T}$ process. Thus $|Q' \cup P'| = |(Q(u) \cup P(u)) \setminus Q| \leq N$ by (13). Finally, we need to ensure Condition c) holds for the x_i'. Now for any A_u, the probability (choosing u uniformly from A_v) that $z \in B(u)$ is at most $\sqrt{\delta}$. Hence the probability that a random A_u makes $\mathbb{P}(z \in B(u)) \geq \sqrt[4]{\delta}$ is at most $\sqrt[4]{\delta}$. Thus Condition c') will fail for some pair (x_i', x_j') with probability at most $n^2 \sqrt[4]{\delta} \leq 0.05$. Hence the bond xy will be open with probability at least $1 - 0.05 - 0.05 = 0.9$. The result now follows provided we choose the parameters n and R so that equations (15)–(20) are satisfied. Lemma 7 implies that for $\eta \leq 1$, $\lambda = \Theta(\eta/\sigma^2)$ and $p_0 = \Theta(1)$. Hence, by (18), $p_2 = \Theta(\eta^2/\sigma^4)$ and (19) is satisfied if we take $n = \lceil 3.05/p_2 \rceil = \Theta(\sigma^4/\eta^2)$. Equation (17) is then satisfied if we take $R = \Omega(\frac{r_1}{\eta} \log(\frac{\sigma^4}{\eta^2}))$, however, we shall in fact set $R = \Theta(\frac{r_1\sigma^2}{\eta} \log(\frac{\sigma^2}{\eta}))$, which is sufficiently
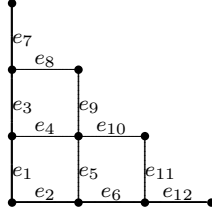
Figure 4: Ordering of the bonds of $\mathbb{Z}^2$ in the proof of Theorem 2.

large since $\sigma^2 > 1$. This will ensure that $C = R/r_1$ can be taken to be independent of r_1 and r_s, and that

$$(R/r_1)^2 > n, \tag{21}$$

which will be used in the proof of Theorem 2. Equations (11)–(13) then imply that $T = \Theta\left(\frac{r_1^2 \sigma^4}{r_s^2 \eta^2} \log^2(\frac{\sigma^2}{\eta})\right)$, $K = \Theta\left(\frac{\sigma^2}{\eta} \log(\frac{r_1 \sigma^2}{r_s \eta})\right)$, and $N = \Theta\left(\frac{r_1^2 \sigma^{10}}{r_s^2 \eta^5} \log^3(\frac{r_1 \sigma^2}{r_s \eta})\right)$. Then $NT = \Theta\left(\frac{r_1^4 \sigma^{14}}{r_s^4 \eta^7} \log^5(\frac{r_1 \sigma^2}{r_s \eta})\right)$, so that Equations (15), (16) and (20) are then satisfied provided

$$\delta = O(r_s^9 r_1^{-9} \eta^{16} \sigma^{-32}).$$

$\square$

Proof of Theorem 2.
Using Theorem 13 we complete the proof. We define the oriented percolation on $\mathbb{Z}^2$. Order the bonds in the first quadrant of $\mathbb{Z}^2$ by their l_1 distance from the origin, and for each distance k, order the bonds at distance k from the origin as

$$(0, k)(0, k + 1) \qquad (0, k)(1, k) \qquad (1, k - 1)(1, k) \qquad \cdots \qquad (k, 0)(k + 1, 0).$$

Suppose the bonds in this order are $\{e_1, e_2, \dots\}$ (see Figure 4). We shall declare some bonds open in such a way that if there is an infinite directed open path in $\mathbb{Z}^2$ from $(0, 0)$ then (with positive probability) there is an infinite path in $\mathcal{G}$. To this end, we shall inductively define for each bond e_i, a subset Q_i of the Poisson process with $Q_{i-1} \subseteq Q_i$, and for each vertex x joined by an open path to the origin in $\mathbb{Z}^2$, a subset P_x of the Poisson process inside C_x.

Initially set $Q_0 = \emptyset$ and set $P_{(0,0)}$ to be any subset of n points of the Poisson process that lie in $C_{(0,0),2}$ and so that each pair of points in $P_{(0,0)}$ are at least $2r_1$ apart. It is easy to check using (21) that such a set exists with high probability (probability 1 if $d > 2$).

Now suppose we have defined the openness of the bonds e_j for $j < i$ and the set Q_i and all relevant P_x. We now consider the bond $e_i = xy$.

  (a) If there is no directed path consisting of open bonds from $(0, 0)$ to x, set xy to be open and $Q_{i+1} = Q_i$.

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(b) If there is a directed open path from $(0,0)$ to x, declare xy to be open if it is open with respect to $P = P_x$ and $Q = Q_i \cap (C_x \cup C_y)$. If xy is open and P_y is not yet defined, set $P_y = P'$ and $Q_{i+1} = Q_i \cup Q'$, otherwise set $Q_{i+1} = Q_i \cup Q' \cup P'$.

Condition (a) is a technical condition which clearly does not affect whether or not $(0,0)$ is in an infinite cluster.

By (b), at each step $|Q_i \cap C_z| \leq kN$ where k is the number of bonds e_j, $j < i$, meeting z. Thus, given the ordering of bonds as above, if e_i is a vertical bond, $|Q_i \cap C_x| \leq 2N$ and $|Q_i \cap C_y| \leq N$, whereas if e_i is a horizontal bond, $|Q_i \cap C_x| \leq 3N$ and $|Q_i \cap C_y| = 0$. Hence the set Q in (b) always satisfies $|Q| \leq 3N$.

There are two bonds $e_i = xy$ with a given value of y, so there are two chances for P_y to be defined in (b). Clearly P_y is defined if and only if y is joined to $(0,0)$ by an open path.

If x is joined to $(0,0)$ by an open path $(0,0) = x_0, \ldots, x_k = x$, then each point in $P_{x_{i+1}}$ is joined by a path in $\mathcal{G}$ from a point in P_{x_i}. Hence there is a path to any point of P_x from one of the n points of $P_{(0,0)}$.

Finally, by Theorem 13, if $\delta < cr_s^9 r_1^{-9} \eta^{16} \sigma^{-32}$ then each bond xy is open with probability bounded below by 0.9 even conditioned on the state of all previous bonds and regions of $\mathbb{R}^2$ that they depend on (the $A(z)$ around the points $z \in Q_i$), except for the other bond starting at x. The two bonds starting at x however can be strongly dependent. Define an oriented *site* percolation on $\mathbb{Z}^2$ by declaring $x \in \mathbb{Z}^2$ to be open if *both* oriented bonds from x are open. This process now stochastically dominates an independent oriented site percolation with site probability 0.8. However $(0,0)$ is then in an infinite cluster with positive probability (see for example [2]). The result now follows. $\qquad \square$

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