Coloring Vertices and Edges of a Path by Nonempty Subsets of a Set

P.N. Balister* E. Győri† R.H. Schelp*‡

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Abstract

A graph G is strongly set colorable if $V(G) \cup E(G)$ can be assigned distinct nonempty subsets of a set of order n, where $|V(G)| + |E(G)| = 2^n - 1$, such that each edge is assigned the symmetric difference of its end vertices. The principal result is that the path $P_{2^{n-1}}$ is strongly set colorable for $n \geq 5$, disproving a conjecture of S.M. Hegde. We also prove another conjecture of Hegde on a related type of set coloring of complete bipartite graphs.

1 Introduction

In a recent article by Hedge [1], he considered coloring certain graphs G, where $|V(G)| + |E(G)| = 2^n - 1$ for some integer n, by nonempty subsets of an n-element set. One primary assignment considered was to assign distinct subsets to the vertices and edges such that each edge is assigned the symmetric difference of its end vertices. Since $|V(G)|+|E(G)| = 2^n-1$ this means all nonempty subsets are used in the assignment. When such an assignment exists it is called a *strong set coloring* of the graph. This assignment is similar to ones frequently studied in coding theory.

One interesting conjecture made in this article was that paths of order 2^{n-1} where $n > 2$ are not strongly set colorable. The primary purpose of this article is to disprove this conjecture for $n \geq 5$. We prove the following theorem.

*Department of Mathematical Sciences, University of Memphis, Memphis TN 38152, USA.

†Alfréd Rényi Institute of Mathematics, Hungarian Academy of Sciences, Budapest, Hungary. Work partially done while the second author was visiting the University of Memphis. Research was partially supported by the EU ToK project FIST 003006, by the OTKA project 48826

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Theorem 1 *The paths P_4 and P_8 are not strongly set colorable while all other paths of the form $P_{2^{n-1}}$ are strongly set colorable.*

An equivalent formulation for a graph to be strongly set colorable and one used throughout this article is the following one. Let G be a connected graph such that $|V(G)| + |E(G)| = 2^n - 1$ for some n. We say that G is *strongly set colorable* if there is a bijection f from $V(G) \cup E(G)$ to the set of nonzero vectors in $\mathbb{F}_2^n$, where $\mathbb{F}_2$ is the field with two elements, such that for every edge $xy \in E(G)$, $f(xy)+f(x)+f(y) = 0$. We regard vectors in $\mathbb{F}_2^n$ as 0-1 sequences of length n, addition in $\mathbb{F}_2^n$ corresponding to componentwise addition modulo 2. The vector v then corresponds to the subset $S_v \subseteq \{1, 2, \ldots, n\}$, where S_v consists of all i such that the ith coordinate of v is 1. In this language the Hedge conjecture says there is no permutation $v_1, v_2, v_3, \ldots, v_{2^n-1}$ of the n-dimensional nonzero vectors of $\mathbb{F}_2^n$ such that $v_i + v_{i+1} + v_{i+2} = 0$ for $i = 1, 3, 5, \ldots, 2^n - 3$. Here v_i represents the set coloring of a vertex when i is odd and of an edge when i is even. These conditions will be referred to as *sum conditions*. When a permutation satisfying the sum conditions exists we will refer to it as *representing vectors in a good permutation* or more briefly as a *good permutation*.

In Section 3 we also briefly discuss some other trees which are strongly set colorable and give a related condition which is of interest in its own right. Finally, in Section 4, we consider a related concept of proper set colorings, and prove a result about proper set colorings of complete bipartite graphs.

2 Results on strong set colorings

Lemma 2 *The paths P_4 and P_8 of orders 4 and 8 are not strongly set colorable.*

Proof. First consider the path P_4 and suppose there is a good permutation of its representing vectors $v_1, \ldots, v_7$. Since for all $n > 1$ the sum of all the non-zero vectors in $\mathbb{F}_2^n$ is zero, $v_1 + v_2 + v_3 + v_4 + v_5 + v_6 + v_7 = 0$. By the sum condition both $v_1 + v_2 + v_3 = 0$ and $v_5 + v_6 + v_7 = 0$, so we have $v_4 = 0$, a contradiction. Therefore P_4 is not strongly set colorable.

Next consider the path P_8. Suppose that there is a good permutation $v_1, \ldots, v_{15}$ of the representing vectors satisfying the sum conditions. First notice that

$$\begin{aligned} v_4 + v_8 + v_{12} &= \sum_{i=1}^{15} v_i - (v_1{+}v_2{+}v_3) - (v_5{+}v_6{+}v_7) - (v_9{+}v_{10}{+}v_{11}) - (v_{13}{+}v_{14}{+}v_{15}) \\ &= 0 - 0 - 0 - 0 = 0. \end{aligned}$$

Claim: The vectors $v_5, v_6, v_7, v_9, v_{10}, v_{11}, v_{13}, v_{14}, v_{15}$ can be written in the form $v_i + v_{4j}$ where $i, j = 1, 2, 3$. We call the forms of these 9 vectors canonical.

Proof of Claim. First we show that these 9 vectors are different from $v_1, v_2, v_3, v_4, v_8, v_{12}$. Suppose that $v_{i_1} = v_{i_2} + v_{4j_2}$. Then adding v_{i_2} to this vector, we obtain that either $v_{4j_2} = 0$ if $i_1 = i_2$ or v_{4j_2} is the third vector out of v_1, v_2, v_3 if $i_1 \neq i_2$, a contradiction in both cases. Likewise $v_{4i_1} \neq v_{i_2} + v_{4j_2}$, otherwise $v_{i_2} = 0$ or the third vector of v_4, v_8, v_{12}. Then, by pigeonhole principle, it is sufficient to prove that these 9 vectors are pairwise distinct. Suppose that $v_{i_1} + v_{4j_1} = v_{i_2} + v_{4j_2}$. If $i_1 = i_2$, $j_1 \neq j_2$, then it follows that $v_{4j_1} = v_{4j_2}$, a contradiction. If $i_1 \neq i_2$ then let i_3 be the third index out of $1, 2, 3$. Then $v_{i_3} = v_{i_1} + v_{i_2} = v_{4j_1} + v_{4j_2}$, which is either 0 if $j_1 = j_2$ or the third vector out of v_4, v_8, v_{12} if $j_1 \neq j_2$, a contradiction, completing the proof of this claim. □

Define the sets $B_1 = \{v_5, v_6, v_7\}, B_2 = \{v_9, v_{10}, v_{11}\}, B_3 = \{v_{13}, v_{14}, v_{15}\}$ as *blocks.*

Claim: For each block, the vectors $v_1, v_2, v_3, v_4, v_8, v_{12}$ appear exactly once in canonical form of the vectors in the block.

Proof of Claim. Let w_1, w_2, w_3 be the vectors in the block, and suppose that, say, v_{i_1} appears in the canonical form of w_1 and w_2, i.e., $w_1 = v_{i_1} + v_{4j_1}$ and $w_2 = v_{i_1} + v_{4j_2}$ where $j_1 \neq j_2$ since $w_1 \neq w_2$. Then $w_3 = w_1 + w_2 = v_{4j_1} + v_{4j_2} = v_{4j_3}$ where v_{4j_3} is the third vector in $\{v_4, v_8, v_{12}\}$, a contradiction. The case when v_{4j_1} appears twice can be settled similarly. □

Having established the above claims we complete the proof of the theorem.

Observe that $v_5 = v_3 + v_4$ by the sum condition. If the canonical form of v_7 contains v_8 then the sum condition implies that v_9 is an element of v_1, v_2, v_3, a contradiction. So v_7 has v_{12} in its canonical form, and since we have symmetry in v_1 and v_2, we may assume that $v_7 = v_1 + v_{12}$ (and so $v_6 = v_2 + v_8$). Then $v_9 = v_1 + v_4$ by the sum condition. As above, v_{11} cannot have v_{12} in its canonical form, and of course, cannot be $v_6 = v_2 + v_8$. Thus $v_{11} = v_3 + v_8$, since if $v_{11} = v_1 + v_8$ then $v_{12} + v_{11} = v_1 + v_4 = v_9 = v_{13}$, a contradiction. Then by the sum condition, $v_{13} = v_3 + v_4 = v_5$, a final contradiction, completing the proof of the theorem. □

Lemma 3 *The path $P_{2^{n-1}}$ of order 2^{n-1} is strongly set colorable for $n = 2$ and $n \geq 5$.*

Proof. The result is clear for $n = 2$ so we assume $n \geq 5$.

The proof is inductive applying the induction separately for n odd and for n even. The idea of the proof is to take four copies of a path P which has been strongly set colored and extend this fixed coloring to a path of order $4|P|$. This involves three basic operations. First, each of the 0-1 vectors of the fixed path using n coordinates is increased to one using $n + 2$ coordinates by adding two new 0-1 coordinates such that the resulting vector is distinguished in each of the four copies of this path. Secondly, the four copies are laid end to end with the order of the second and fourth copies reversed from the first and third. Lastly, three new vectors each with zeros in the first $n - 1$ coordinates and differing only in at least one of the last two 0-1 coordinates are inserted between the first path and the second, the second and the third, and the third and the fourth.

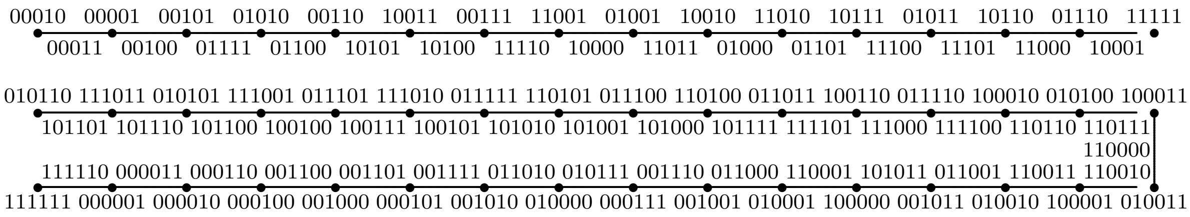

Figure 1: Example set colorings of path P_{2^n-1} for $n = 5$ and $n = 6$, sets given as 0-1 sequences for each edge and vertex.

In order to accomplish what has just been described in a precise formal way we first introduce some notation. If v is an n-dimensional vector (0-1 sequence) then $v0$ and $v1$ are the $(n+1)$-dimensional vectors obtained from v by extending it with a coordinate 0 and 1, respectively. Similarly, say, $v11$ is the $(n+2)$-dimensional vector obtained from v by extending it with two coordinates 1. To avoid confusion, the n-dimensional zero vector will be denoted by 0_n.

Figure 1 shows that there are good permutations for both $n = 5$ and $n = 6$. It is easy to verify the correctness of these examples. These examples will anchor the inductive proof that we give separately for n odd and for n even.

To finish the proof of the theorem, we prove for that if P_{2^n-1} is strongly set colorable for $n \geq 5$ then $P_{2^{n+1}}$ is strongly set colorable.

Thus assume that $v_1, v_2, \ldots, v_{2^n-1}$ is a desired good permutation of the n-dimensional nonzero 0-1 vectors. Then it is a somewhat tedious, but straightforward to verify that the following sequence is a desired permutation of the $(n+2)$-dimensional nonzero vectors satisfying the sum conditions. (The extension is 4-periodic in the blocks of $2^n - 1$ vectors but requires certain interchanges of the first or last two vectors to fulfill the sum conditions). We provide the lengthy sequence to clarify the desired resulting good permutation. (Vectors written vertically, bold entries indicate entries that do not fit into the 4-periodic pattern.)

	v_1	v_2	v_3	v_4	v_5	v_6	v_7		v_{2^n-8}	v_{2^n-7}	v_{2^n-6}	v_{2^n-5}	v_{2^n-4}	v_{2^n-3}	v_{2^n-2}	v_{2^n-1}
	0	0	0	0	0	0	0	...	0	0	0	0	0	0	0	0
	0	0	0	0	0	0	0		0	0	0	0	0	0	0	0
$\mathbf{0_n}$	v_{2^n-1}	v_{2^n-2}	v_{2^n-3}	v_{2^n-4}	v_{2^n-5}	v_{2^n-6}	v_{2^n-7}		v_8	v_7	v_6	v_5	v_4	v_3	$\mathbf{v_1}$	$\mathbf{v_2}$
1	1	0	1	0	1	0	1	...	0	1	0	1	0	1	**1**	**0**
1	1	1	0	1	1	1	0		1	1	1	0	1	1	**0**	**1**
$\mathbf{0_n}$	$\mathbf{v_2}$	$\mathbf{v_1}$	v_3	v_4	v_5	v_6	v_7		v_{2^n-8}	v_{2^n-7}	v_{2^n-6}	v_{2^n-5}	v_{2^n-4}	v_{2^n-3}	$\mathbf{v_{2^n-1}}$	$\mathbf{v_{2^n-2}}$
1	**1**	**0**	1	1	0	1	1	...	1	0	1	1	1	0	**1**	**1**
0	**1**	**1**	0	1	1	1	0		1	1	1	0	1	1	**0**	**1**
$\mathbf{0_n}$	$\mathbf{v_{2^n-2}}$	$\mathbf{v_{2^n-1}}$	v_{2^n-3}	v_{2^n-4}	v_{2^n-5}	v_{2^n-6}	v_{2^n-7}		v_8	v_7	v_6	v_5	v_4	v_3	v_2	v_1
0	**1**	**0**	1	1	0	1	1	...	1	0	1	1	1	0	1	1
1	**0**	**1**	1	0	1	0	1		0	1	0	1	0	1	0	1

This completes the inductive step and the proof of the theorem. □

The reader should note that the interchanges required in the constructed sequence in the proof just given makes this construction fail for $n = 2$, so that P_2 strongly set colorable does not imply P_8 is strongly set colorable.

The general strategy used in the proof of the last theorem can be applied beginning with a strongly set colorable bipartite graph in place of a path. The interchange idea is not needed in this case.

Theorem 4 *Let G be a strongly set colorable bipartite graph with color classes X, Y and edge set E. Let G_1, G_2, G_3, G_4 be four disjoint copies of G with color classes X_1, Y_1, X_2, Y_2, X_3, Y_3, X_4, Y_4 and edge sets E_1, E_2, E_3, E_3, respectively. Let G_0 denote the graph obtained from the disjoint union of the graphs G_1, G_2, G_3, G_4 by adding edges e_1, e_2, e_2 with the following properties:*

1. *each e_i joins two copies of the same vertex;*
2. *one of the following three possibilities occurs:*
   (a) *the edges join X_1 and X_2, X_1 and X_3, X_1 and X_4, respectively; or*
   (b) *the edges join X_1 and X_2, X_2 and X_3, Y_3 and Y_4, respectively; or*
   (c) *the edges join X_1 and X_2, Y_2 and Y_4, Y_1 and Y_3, respectively.*

Then G_0 is strongly set colorable.

Proof. Consider the same vector coloring of G_1, G_2, G_3 and G_4. Then extend the vectors in X_1, Y_1, and E_1 with 00, the vectors in X_3, Y_4, and E_2 with 01, the vectors in X_2, Y_3, and E_4 with 10, the vectors in X_4, Y_2, and E_3 with 11. If we color the edges e_1, e_2, e_3 according to the sum condition, then it is easy to verify that we obtain a desired vector coloring of G_0, completing the proof. □

Symmetric possibilities of those listed in the last theorem are clearly strongly set colorable as well. It is noted in [1] that graphs which are strongly set colorable cannot have it even degree vertices covered by two edges. However this condition is not sufficient for graphs to be strongly set colorable. For example, P_8 is not strongly set colorable but does not have its six even degree vertices covered by two edges. It seems to be difficult to even characterize which trees are strongly set colorable.

3 Strong set colorings of other trees

The binary tree has a particularly nice set coloring. To ensure that we have exactly 2^{n-1} vertices, we add one additional leaf to the root of the tree. The coloring is then given at each

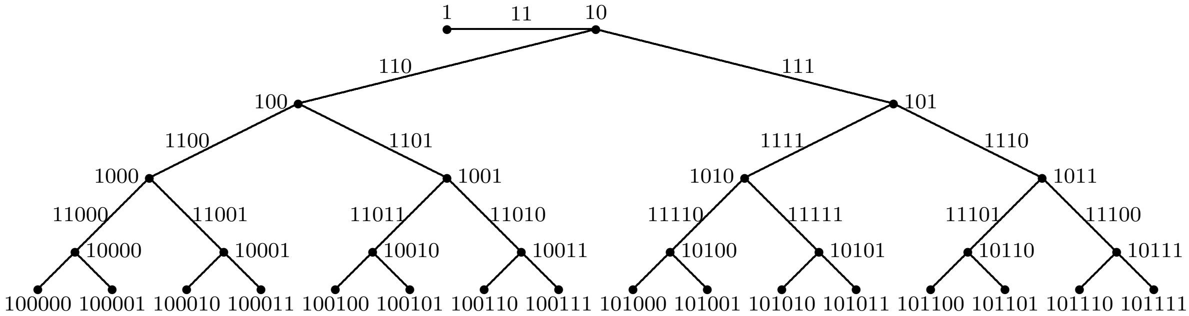

Figure 2: Coloring of the binary tree. All 0-1 sequences should be padded on the left by the appropriate number of zeros.

level by assigning vertices the binary numbers from $2^{\ell+1}$ to $3.2^{\ell}-1$ in turn, see Figure 2. The edges the are colored with the numbers 3.2^{ℓ} to $2^{\ell+2}-1$ in the order of a grey code.

Another interesting possible construction is to take a strong set coloring of G and extend it to a new graph G' obtained from G by adding 2^n pending edges. The color of each of the original edges and vertices of G is obtained by appending 0 to the original color. New edges e_i get color $1p_i$ and new vertices u_i get color $1q_i$ where the pendent edge $e_i = u_iv_i$ is joined to a vertex v_i of color $0v_i$. We need $v_i = p_i - q_i$ for this to give a strong coloring of G', and the set of all p_i and q_i needs to exhaust $\mathbb{F}_2^n$. Note that some of the v_i's may be the same since we can attach several pendent edges to the same vertex. We need $\sum v_i = 0$ for such p_i and q_i to exist (since $0 = \sum p_i + \sum q_i = \sum v_i$). It appears that this is sufficient, so we make the following conjecture.

Conjecture 1 *Given 2^{n-1} non-zero (not necessarily distinct) vectors $v_1, \ldots, v_{2^{n-1}} \in \mathbb{F}_2^n$, $n \geq 2$, with $\sum_{i=1}^{2^{n-1}} v_i = 0$, there exists a partition of $\mathbb{F}_2^n$ into pairs of vectors $\{p_i, q_i\}$, $i = 1, \ldots, 2^{n-1}$ such that for all i, $v_i = p_i - q_i$.*

This conjecture, which appears to be of interest in its own right, is true for $n \leq 5$. It is also true if at least half of all the vectors v_i are the same and each vector occurs an even number of times.

Theorem 5 *Given 2^{n-1} non-zero vectors $v_1, \ldots, v_{2^{n-1}} \in \mathbb{F}_2^n$, $n \geq 2$, with $v_1 = v_2 = \cdots = v_{2^{n-2}}$ and $v_{2i+1} = v_{2i+2}$ for all $i = 0, \ldots, 2^{n-2}-1$, there exists a partition of $\mathbb{F}_2^n$ into pairs of vectors $\{p_i, q_i\}$, $i = 1, \ldots, 2^{n-1}$ such that for all i, $v_i = p_i - q_i$.*

Proof. Without loss of generality assume $v_1 = 00\ldots01 \in \mathbb{F}_2^n$. Suppose we have chosen p_j and q_j for some pairs $j = 2i+1, 2i+2$ where $v_j \neq v_1$ so that the vectors that have been used form a union of pairs $\{u0, u1\}$, $u \in S \subseteq \mathbb{F}_2^{n-1}$. Suppose $v_{2i+1} = v_{2i+2} \neq v_1$ is another pair of vectors for which p_j and q_j have not yet been assigned. Write $v_{2i+1} = v_{2i+2} = va$ where $v \in \mathbb{F}_2^{n-1}$ and $a \in \{0, 1\}$. Since we have chosen fewer than half of all the p_j and q_j,

$|S| < 2^{n-2}$. Thus by the pigeonhole principle, there exists a pair $p, q \notin S$ with $p - q = v$. Now choose $p_{2i+1} = pa$, $q_{2i+1} = q0$, $p_{2i+2} = p\bar{a}$, and $q_{2i+2} = q1$, where $\bar{a} = 1 - a$. Then $v_{2i+1} = p_{2i+1} - q_{2i+1}$ and $v_{2i+2} = p_{2i+2} - q_{2i+2}$, and the chosen vectors form a union of pairs $\{u0, u1\}$ with $u \in S' = S \cup \{p, q\}$. Repeat this process until we have assigned vectors p_j and q_j for all $v_j \neq v_1$. Finally assign for each $v_j = v_1$ one of the remaining $p \notin S$ and set $p_j = p1$ and $q_j = p0$ so that $v_j = p_j - q_j$ for these values of j as well. This gives the required partition of $\mathbb{F}_2^n$ into pairs. □

4 Proper set colorings of complete bipartite graphs

In [1], a *proper set coloring* of a graph G is defined as an assignment of subsets of an n-element set to the vertices and edges of G so that

1. each edge receives the symmetric difference of the sets assigned to its end vertices;
2. distinct vertices receive distinct sets;
3. distinct edges receive distinct non-empty sets; and
4. every non-empty set appears on some edge.

Note that the same set can occur on both an edge and a vertex. Clearly, if a proper set coloring of G is to exists we need $|E(G)| = 2^n - 1$.

The following theorem appears as a conjecture in [1].

Theorem 6 *If the complete bipartite graph $K_{s,t}$ has a proper set coloring then either $s = 1$ or $t = 1$.*

Writing $S_1, \dots, S_s$ and $T_1, \dots, T_t$ as the sets assigned to the vertices of $K_{s,t}$, this theorem is an immediate consequence of the following.

Theorem 7 *Assume $S_1, \dots, S_s$ and $T_1, \dots, T_t$ are subsets of $\{1, \dots, n\}$ and the symmetric differences $S_i \oplus T_j$, $i = 1, \dots, s$, $j = 1, \dots, t$, are non-empty and represent every non-empty subset of $\{1, \dots, n\}$ exactly once (so in particular $st = 2^n - 1$). Then either $s = 1$ or $t = 1$.*

Proof. Consider for any $X \subseteq \{1, \dots, n\}$ the quantities $\hat{S}_X = \sum_{i=1}^{s} (-1)^{|X \cap S_i|}$ and $\hat{T}_X = \sum_{j=1}^{t} (-1)^{|X \cap T_j|}$. (These are Fourier transforms over $\mathbb{F}_2^n$ of the indicator functions giving the sets S_i and T_j respectively.) Now

$$\hat{S}_X \hat{T}_X = \sum_{i,j} (-1)^{|X \cap S_i| + |X \cap T_j|} = \sum_{Y \neq \emptyset} (-1)^{|X \cap Y|}$$

since $|X \cap S_i| + |X \cap T_j| \equiv |X \cap (S_i \oplus T_j)| \bmod 2$ and the $S_i \oplus T_j$ range over all non-empty subsets of $\{1, \ldots, n\}$. Now if $X \neq \emptyset$, then the number of (possibly empty) subsets $Y \subseteq \{1, \ldots, n\}$ with $|X \cap Y|$ is even is the same as the number with $|X \cap Y|$ odd. Indeed, if $k \in X$ we can pair up the sets Y with $k \notin Y$ with $Y' = Y \cup \{k\}$ and then $|X \cap Y|$ and $|X \cap Y'|$ have different parities. Thus

$$\hat{S}_X \hat{T}_X = \sum_{Y \neq \emptyset} (-1)^{|X \cap Y|} = \Big(\sum_{Y} (-1)^{|X \cap Y|} \Big) - (-1)^{|X \cap \emptyset|} = 0 - 1 = -1.$$

But $\hat{S}_X, \hat{T}_X \in \mathbb{Z}$, so $\hat{S}_X, \hat{T}_X \in \{-1, 1\}$ for $X \neq \emptyset$. Now consider

$$\sum_X \hat{S}_X^2 = \hat{S}_\emptyset^2 + \sum_{X \neq \emptyset} \hat{S}_X^2 = s^2 + (2^n - 1).1.$$

We can also write this as

$$\sum_X \hat{S}_X^2 = \sum_X \sum_{i,j=1}^{s} (-1)^{|X \cap S_i|} (-1)^{|X \cap S_j|} = \sum_{i,j=1}^{s} \sum_X (-1)^{|X \cap (S_i \oplus S_j)|}.$$

But the last sum over X is zero unless $S_i \oplus S_j = \emptyset$, i.e., unless $S_i = S_j$. But if $S_i = S_j$ then $S_i \oplus T_1 = S_j \oplus T_1$, so $i = j$ since the $S_k \oplus T_l$ are distinct. Hence

$$\sum_X \hat{S}_X^2 = \sum_{i=1}^{s} \sum_X (-1)^{|X \cap \emptyset|} = 2^n s.$$

(This is just a special case of Parseval's theorem for Fourier transforms.) Thus $2^n s = s^2 + (2^n - 1)$ and so

$$(s-1)(s-(2^n-1)) = s^2 - 2^n s + (2^n - 1) = 0.$$

Hence either $s = 1$, or $s = 2^n - 1$. But if $s = 2^n - 1$ then $t = 1$ since $st = 2^n - 1$. $\square$

References

[1] S.M. Hegde, Set Colorings of Graphs, *preprint*.