A note on Ramsey size-linear graphs

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Abstract

We show that if G is a Ramsey size-linear graph and $x, y \in V(G)$ then if we add a sufficiently long path between x and y we obtain a new Ramsey size-linear graph. As a consequence we show that if G is any graph such that every cycle in G contains at least four consecutive vertices of degree 2 then G is Ramsey size-linear.

If G is a graph, write $n(G) = |V(G)|$ for the number of vertices and $e(G) = |E(G)|$ for the number of edges of G.

It is well known that the Ramsey number $r(K_3, T) = 2e(T) + 1$ for any tree T. In the early 1980's Harary asked if $r(K_3, H) \leq 2e(H) + 1$ for every graph H. An upper bound was given in [4], later improved by Sidorenko [6], and then in 1993 the "Harary bound" was shown to hold by Sidorenko [7]. This motivated the following definition, which is equivalent to the one introduced in [5].

Definition 1. *A graph G is* Ramsey size-linear *if there is a constant C_G such that for any graph H the Ramsey number $r(G, H)$ is bounded above by $C_G e(H) + n(H)$.*

Note that this implies $r(G, H)$ is bounded above by the linear function $(C_G + 2)e(H)$ when H has no isolated vertices. In [5] the following results were proved.

1. Any connected graph with $e(G) \leq n(G) + 1$ is Ramsey size-linear.
2. Any graph with $e(G) \geq 2n(G) - 2$ is *not* Ramsey size-linear.
3. Any graph of the form $K_1 + T$ is Ramsey size-linear, where T is a tree (or forest) and $K_1 + T$ is the graph obtained by joining a single vertex v to every vertex of T.
4. Any (bipartite) graph with extremal number $\mathrm{ext}(G, n) = O(n^{3/2})$ is Ramsey size-linear.
5. If G is obtained from $G_1 \cup G_2$ by identifying a vertex of G_1 with a vertex of G_2 and if G_1 and G_2 are Ramsey size-linear then so is G.

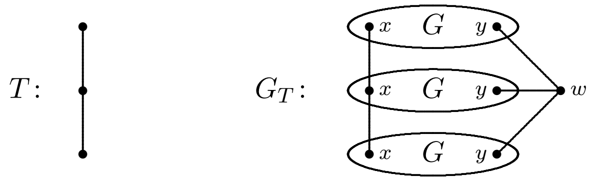

Figure 1: The graph G_T.

It is also clear that any subgraph of a Ramsey size-linear graph is also Ramsey size-linear.

As a consequence of Property 2, the graph K_4 is not Ramsey size-linear. In particular it has been shown that

$$C(n/\log n)^{5/2} \leq r(K_4, K_n) \leq C'n^3/(\log n)^2.$$

The lower bound is due to Spencer [8] using the Lovász Local Lemma, and the upper bound is due to Ajtai, Komlós and Szemerédi [1]. Erdős [3] asked for a proof or disproof that $r(K_4, K_n) \geq n^3/(\log n)^c$ for some c, offering \$250 for a solution.

It is therefore of interest whether any graph G which is a topological K_4 is Ramsey size-linear. In particular, is the graph G formed by subdivision of an edge of K_4 one or more times Ramsey size-linear? In this note we show that if G is any Ramsey size-linear graph and $x, y \in V(G)$ then we can join x and y by a path of suitable length so that the resulting graph is Ramsey size-linear. Hence for any graph G it is possible to subdivide the edges so that the resulting graph is Ramsey size-linear. In particular, for K_4, subdividing one of the edges four times is sufficient. It is an open question as to whether K_4 with an edge subdivided just once is Ramsey size-linear.

Assume T is a tree (or forest) and G is any graph. Let x and y be vertices of G (possibly equal) and define a graph G_T as follows. Let $x_1, \ldots, x_t$ be the vertices of T. Take t copies of G and fix in each of them a vertex corresponding to x and a vertex corresponding to y. Now join x in the i'th copy to the x in the j'th copy if $x_i x_j \in E(T)$. Join y in each copy to a single new vertex w. The resulting graph will be G_T (see Figure 1).

Theorem 1. *Assume T is a forest, G is Ramsey size-linear and $x, y \in V(G)$ (possibly equal). Let G_T be defined as above. Then G_T is Ramsey size-linear. Indeed we can take $C_{G_T} = C_G + 2 + 2(n(T) - 1)n(G)$.*

Proof. We prove the result by induction on $n(H)$. The result clearly holds for $n(H) = 1$ since then $r(G_T, H) = 1$. Adding an isolated vertex to H can increase $r(G_T, H)$ by at most 1. Hence we may assume H has no isolated vertex. Let $v \in H$ be a vertex of minimum degree $\delta = \delta(H)$ and assume the result holds for $H - v$. Hence if we have a 2-coloring of K_n without red G_T, and $n \geq C_{G_T}(e(H) - \delta) + (n(H) - 1)$, then it must contain a blue H_1 isomorphic to $H - v$. Let N be the set of vertices of H_1 corresponding to the neighbors of v in H. Let S be the set of vertices of K_n that do not lie in H_1. If a vertex $u \in S$ is joined to all the vertices in N by blue edges then adding u to H_1 gives a blue H, hence we may assume every vertex of S has at least one red edge to N. For each $u \in S$ pick one such edge. This partitions S as a disjoint union $\cup_{w \in N} S_w$ according to the vertex $w \in N$ this chosen edge is incident to.

Now use the fact that $r(G,H) \leq (C_G + 2)e(H)$ to find many vertex disjoint copies of red G's in S. We can find by induction a total of at least $s = (|S| - (C_G + 2)e(H))/n(G)$ such copies since S spans no blue H. Let X_w be the set of the x's of these G's, such that the corresponding y's are in S_w. Hence $\sum_{w \in N} |X_w| \geq s$.

If $s > (r(T,H) - 1)|N|$ then there must be some $w \in N$ such that $|X_w| \geq r(T,H)$. Since the subgraph spanned by X_w contains no blue H, it must contain a red T. This red T together with the graphs G it meets and the vertex w form a red G_T.

Now $r(T,H) \leq r(T, K_{n(H)}) = (n(T) - 1)(n(H) - 1) + 1$ (see [2]). Hence it is sufficient if $s > (n(T) - 1)(n(H) - 1)|N|$. However, $n(H)|N| \leq 2e(H)$, so it is enough that $s > 2(n(T) - 1)e(H)$, or $|S| > (C_G + 2 + 2(n(T) - 1)n(G))e(H)$. Since $n = |S| + n(H) - 1$, the result follows with $C_{G_T} = C_G + 2 + 2(n(T) - 1)n(G)$. □

Corollary 2. *If G is Ramsey size-linear and x and y are two vertices in the same component of G (possibly the same vertex), then the graph G' obtained by adding a path (cycle if $x = y$) of length r between x and y is also Ramsey size-linear provided $r \geq d(x,y) + 3$, where $d(x,y)$ is the distance between x and y in G. If x and y lie in different components of G then G' is Ramsey size linear for any $r \geq 0$.*

Proof. Let T be a path of length $r - d(x,y) - 2 \geq 1$. Then G_T contains a subgraph isomorphic to G' by taking one copy of G joined to one end of T, with x and y joined by T, a path of length $d(x,y)$ in the copy of G at the other end of T and then a path of length 2 via w. The result follows since a subgraph of a Ramsey size-linear graph is Ramsey size-linear. If x and y belong to distinct components of G then the graph obtained by identifying them is also Ramsey size-linear. Adding a path $x \dots x'$ of length r to x first and identifying x' and y now proves the second part. □

The graph K_4 with an edge deleted is Ramsey size-linear by Property 3 above. Taking xy as the deleted edge and applying Corollary 2 shows that K_4 with an edge subdivided four times is Ramsey size-linear.

Corollary 3. *If G is a graph such that every cycle in G contains at least four consecutive vertices of degree 2, then G is Ramsey size-linear.*

Proof. By removing suspended paths of length 5 from G we can obtain a graph T without cycles, i.e., a forest. Now $K_1 + T$ is Ramsey size-linear and given any $x, y \in V(T)$ there is a path of length at most 2 joining x and y in $K_1 + T$. Applying Corollary 2 we can add paths of length $5 \geq d(x,y) + 3$ to $K_1 + T$, thus replacing the suspended paths we removed from G. (Note that x may be equal to y.) Finally, removing the vertex of K_1 gives the graph G. □

It is an interesting question as to how much Corollary 2 can be improved. As a special case, we have the following important question.

Question 1. *Is the graph G obtained from K_4 by subdividing one of its edges once Ramsey size-linear?*

Also one can ask a more general question.

Question 2. *Is it always the case that if G is Ramsey size-linear and G' is obtained from G by joining two vertices by a path of length 2 then G' is necessarily Ramsey size-linear?*

If the answer to this last question is Yes, then any graph is Ramsey size-linear unless it contains a subgraph H with no cut vertex and $\delta(H) \geq 3$. On the other hand, any graph H with no cut vertex and $\delta(H) \geq 3$ cannot be constructed by joining vertices of a smaller graph by paths of length 2 or by identifying vertices of two smaller as in Property 5 above. We can therefore also ask the following question.

Question 3. *Is it always the case that if G has no cut vertex and the minimum degree of G is at least 3 then G is not Ramsey size-linear?*

If the answer to the last two questions is Yes, then we would obtain a complete characterization of Ramsey size-linear graphs.

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