Mono-multi bipartite Ramsey numbers, designs, and matrices

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Abstract

Eroh and Oellerman defined $BRR(G_1, G_2)$ as the smallest N such that any edge coloring of the complete bipartite graph $K_{N,N}$ contains either a monochromatic G_1 or a multicolored G_2. We restate the problem of determining $BRR(K_{1,\lambda}, K_{r,s})$ in matrix form and prove estimates and exact values for several choices of the parameters. Our general bound uses Füredi's result on fractional matchings of uniform hypergraphs and we show that it is sharp if certain block designs exist. We obtain two sharp results for the case $r = s = 2$: we prove $BRR(K_{1,\lambda}, K_{2,2}) = 3\lambda - 2$ and that the smallest n for which any edge coloring of $K_{\lambda,n}$ contains either a monochromatic $K_{1,\lambda}$ or a multicolored $K_{2,2}$ is λ^2.

1 Introduction

Problems concerning configurations which must be present in any edge coloring of a complete graph emerged from a classical paper of Erdős and Rado [6]. The most frequently studied (canonical) configurations of [6] are the monochromatic and the multicolored. The

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former goes back to the classical paper of Ramsey [13], (see also [10]) and the latter probably reappeared first in a paper of Erdős, Simonovits and Sós [7]. It seems that the variant when either monochromatic or multicolored configurations are sought — perhaps one may call it *mono-multi Ramsey numbers* — was mentioned first in [1], although the concept (for arithmetic progressions) already appeared in [5]. Without completeness, we point out some recent references on the subject: [8], [11], [16].

Eroh and Oellerman [8] considered the mono-multi Ramsey numbers for bipartite graphs: they defined $BRR(G_1, G_2)$ as the smallest N such that any edge coloring of $K_{N,N}$ contains either a monochromatic G_1 or a multicolored G_2. They proved that this number exists if and only if G_1 is a star or G_2 is a star forest. Among many results they proved is that $BRR(K_{1,\lambda}, K_{r,s}) = O(\lambda)$ for fixed r, s (in [8] Theorem 2). In particular, for $r = s = 2$ (i.e., when G_2 is a 4-cycle) they showed that $3\lambda - 2 \leq BRR(K_{1,\lambda}, K_{2,2}) \leq 6\lambda - 8$ (see [8] p.71). We shall prove (Theorem 5) that the lower bound is the correct value.

We found it convenient to restate the problem in terms of matrices. Given positive integers λ, r, s, find the smallest $n = n_\lambda(r, s)$ with the following property: in any $n \times n$ matrix A either some entry is repeated at least λ times in some row or column, or there is an $r \times s$ submatrix B in A whose elements are all distinct. Since only equality among elements of A matters, we specify entries as most convenient in particular situations. We also formulate an asymmetric version of the problem: given positive integers λ, r, s and m, find the smallest $n = n_\lambda(r, s; m)$ such that in any $m \times n$ matrix A either some entry is repeated at least λ times in some row or column or there is an $r \times s$ submatrix B in A whose elements are all distinct.

Observe that for $m \leq (r-1)(\lambda-1)$, $n_\lambda(r, s; m)$ is undefined: consider any number of columns, each filled with $r-1$ symbols repeated $\lambda - 1$ times (using distinct symbols in distinct columns). However, Theorem 3 shows that for $m > (r-1)(\lambda-1)$ the function $n_\lambda(r, s; m)$ exists. It is worth noting that $n_\lambda(r, r; m)$ becomes the mono-multi Ramsey number of unbalanced bipartite graphs, i.e. for any λ, r and $m > (r-1)(\lambda-1)$, there exists a least integer $n = n_\lambda(r, r; m)$ such that every edge coloring of $K_{m,n}$ contains either a monochromatic $K_{1,\lambda}$ or a multicolored copy of $K_{r,r}$.

In Section 4 we determine the exact value of $n_\lambda(3, s; m)$ for infinitely many appropriate values of m, λ, and s (Theorem 9). The particular cases $n_3(3, 3; 5) = n_3(3, 3; 6) = 71$ settled in Theorem 8 translate into the following mono-multi Ramsey result: every edge coloring of $K_{5,71}$ contains either a monochromatic $K_{1,3}$ or a multicolored $K_{3,3}$, moreover this is not true for $K_{6,70}$.

Notice that we can assume $r, s \geq 2$ since obviously

$$n_\lambda(1, s) = (s-1)(\lambda-1) + 1, \qquad n_\lambda(r, 1) = (r-1)(\lambda-1) + 1.$$

An *extremal matrix* is an $n \times n$ (or $m \times n$) matrix A with $n = n_\lambda(r, s) - 1$ (or $n = n_\lambda(r, s; m) - 1$) such that every element of A is repeated at most $\lambda - 1$ times in rows and in columns and A has no $r \times s$ submatrix with distinct elements. Matrices obtained by row or column permutations or by transformations on the values of the entries preserving equality are considered to be the same.

We shall prove (Theorem 5) that $n_\lambda(2,2) = 3\lambda - 2$, implying that the lower bound in [8] is the true value of $BRR(K_{1,\lambda}, K_{2,2})$. For the asymmetric version, we show (Theorem 7) that $n_\lambda(2,2;\lambda) = \lambda^2$ and the extremal matrix is unique. This implies that for any λ and $n \geq \lambda^2$ every edge coloring of $K_{\lambda,n}$ contains either a monochromatic $K_{1,\lambda}$ or a multicolored $K_{2,2}$.

The following lemma is important to establish a general upper bound.

Lemma 1. *Suppose that A is an $r \times n$ matrix such that its columns have no repeated elements and its rows have elements repeated less than λ times. Furthermore, A has no $r \times s$ submatrix with all distinct entries. Then*

$$n \leq (r^2 - r + 1)(s-1)(\lambda - 1). \tag{1}$$

The inequality is sharp if $r = 2$ or when $r - 1$ is a power of a prime.

Proof. Consider the hypergraph $\mathcal{H}$ whose vertex set is the set of all (distinct) entries in A and whose edges are defined by the entries of the columns of A. Each edge has r vertices (i.e., $\mathcal{H}$ is r-uniform). Clearly, the maximum degree, $\Delta = \Delta(\mathcal{H})$, satisfies

$$\Delta \leq r(\lambda - 1)$$

since any entry in any row of A is repeated at most $\lambda - 1$ times. Furthermore, $\nu(\mathcal{H})$, the maximum number of pairwise disjoint edges of $\mathcal{H}$, satisfies

$$\nu(\mathcal{H}) \leq s - 1$$

because A has no $r \times s$ submatrix with distinct entries.

Observe that the assignment of $\frac{1}{\Delta}$ to all edges of $\mathcal{H}$ is a fractional matching, i.e., the sum of the weights of the edges incident with any given vertex is at most 1. Thus for the maximum total sum, $\nu^*(\mathcal{H})$, of the edge weights in a fractional matching of $\mathcal{H}$ we have

$$\frac{|\mathcal{E}(\mathcal{H})|}{\Delta} \leq \nu^*(\mathcal{H}).$$

Füredi proves in [9] that

$$\nu^*(\mathcal{H}) \leq \frac{r^2 - r + 1}{r}\nu(\mathcal{H}),$$

with strict inequality if $PG(2, r-1)$, the finite projective plane of order $r - 1$, does not exist. This result combined with the inequalities above give the bound in the lemma as follows:

$$n = |\mathcal{E}(\mathcal{H})| \leq \nu^*(\mathcal{H})\Delta \leq \frac{r^2 - r + 1}{r}\nu(\mathcal{H})\, r(\lambda - 1) \leq (r^2 - r + 1)(s-1)(\lambda - 1).$$

To see that (1) is sharp, for $i = 1, 2, \ldots, s-1$, let M_i be the $r \times (r^2 - r + 1)$ matrix of a projective plane of order $r - 1$ with points $p_i^1, p_i^2, \ldots, p_i^{r^2-r+1}$, where the columns represent the lines of the plane. A theorem of Singer [15] says that a projective plane over a finite field forms a cyclic design. Hence such a matrix representation exists with no row containing a point more than once. Also, if $r = 2$, we can take

$$M_i = \begin{pmatrix} p_i^1 & p_i^2 & p_i^3 \\ p_i^2 & p_i^3 & p_i^1 \end{pmatrix} .$$

The matrix A is obtained by the (horizontal) concatenation of the M_is with distinct entries for $i = 1, \dots, s-1$ and then the repetition of each column $\lambda - 1$ times. Note that for the hypergraph $\mathcal{H}$ defined by A, each inequality including the one in Füredi's result becomes equality. □

It is worth noting that the case $r = 2$ of Lemma 1 can be proved without Füredi's theorem. It follows immediately from Shannon's theorem stating that the chromatic index of a multigraph is at most 3/2 times its maximum degree [14]. (Also, for $r = 2$ one can refer to Lovász' theorem [12]: $\nu^*(G) \le \frac{3}{2}\nu(G)$, that has a significantly easier proof than its generalization by Füredi). For $\lambda = 2$ Lemma 1 becomes

Proposition 2. *If $r = 2$ or $r - 1$ is a power of prime, then*

$$n_2(r, s; r) = (r^2 - r + 1)(s - 1) + 1. \quad \square$$

The significance of Lemma 1 is that one can apply it to get a general upper bound (Theorem 3) for the function $n_\lambda(r, s; m)$ which turns out to be sharp in infinitely many cases (see Section 4). We found it amusing to combine the factorization of K_6 with Fano planes to give an extremal 6×70 matrix showing that the upper bound $n_3(3, 3; 6) \le 71$ is in fact an equality (see Theorem 8 below). We were somewhat disappointed when it turned out that the 170 blocks of the unique 3-$(17, 8, 14)$ design (due to Brouwer [3]) cannot be partitioned into disjoint pairs. This fact and Füredi's aforementioned theorem imply the nonexistence of a 17×4760 extremal matrix, and thus the bound $n_9(3, 9; 17) \le 4761$ is not tight.

2 A general bound

Here we give an upper bound on $n_\lambda(r, s; m)$ based on Lemma 1. Given an m-tuple $a = (a_1, \dots, a_m)$, let $p(a, r)$ be the number of subsets $S \subseteq \{1, \dots, m\}$ of size r such that the r elements a_i, $i \in S$, are all distinct. Set $p(\lambda, m, r)$ to be the minimum value of $p(a, r)$ over all m-tuples a for which every element of a is repeated less than λ times in a. It is well known — and easy to prove — that $p(\lambda, m, r)$ is achieved for the coarsest partition of $[m]$, i.e., if $m = q(\lambda - 1) + t$, $0 \le t < \lambda - 1$, then a contains q entries repeated $\lambda - 1$ times and one entry repeated t times. Thus

$$p(\lambda, m, r) = \tbinom{q}{r}(\lambda - 1)^r + \tbinom{q}{r-1}(\lambda - 1)^{r-1}t \quad \text{where} \quad m = (\lambda - 1)q + t,\ 0 \le t < \lambda - 1.$$

Theorem 3.

$$n_\lambda(r, s; m) \le 1 + \left\lfloor \tbinom{m}{r}(r^2 - r + 1)(s - 1)(\lambda - 1)p(\lambda, m, r)^{-1} \right\rfloor .$$

Proof. Assume that A is an $m \times n$ extremal matrix. Every column of A has at least $p(\lambda, m, r)$ r-tuples of distinct elements, therefore A has at least $p(\lambda, m, r)n$ r-tuples of distinct elements in its columns. By the pigeonhole principle, at least

$$t = p(\lambda, m, r)n/\binom{m}{r} \tag{2}$$

of these r-tuples are placed along the same set of r rows. By Lemma 1, applied to the corresponding $r \times n_0$ submatrix of these r-tuples, we obtain

$$p(\lambda, m, r)n/\binom{m}{r} \leq n_0 \leq (r^2 - r + 1)(s - 1)(\lambda - 1)$$

implying the required inequality. □

We state one special case of Theorem 3 when the function p has a convenient form. If $m = (r-1)(\lambda-1)+1$, then $p(\lambda, m, r) = (\lambda-1)^{r-1}$ and we obtain

Corollary 4. *For* $m = (r-1)(\lambda-1)+1$,

$$n_\lambda(r, s; m) \leq 1 + \left\lfloor \binom{m}{r}(r^2 - r + 1)(s-1)(\lambda-1)^{2-r} \right\rfloor \tag{3}$$

□

We shall see some examples in Section 4 where (3) is an equality.

3 Two by two submatrices

Theorem 5. *For every* $\lambda \geq 2$, $n_\lambda(2, 2) = 3\lambda - 2$.

Proof. The Latin square

$$L_1 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{pmatrix}$$

has no repeated entries in its rows or columns, and does not contain a 2×2 submatrix of distinct elements. On the other hand, it is easy to see that any 4×2 submatrix of a 4×4 matrix with distinct entries in each row and column has a 2×2 submatrix of distinct elements. Thus the case $\lambda = 2$ is trivial.

Note that by repeating each row of L_1 then each column $(\lambda - 1)$ times, we obtain a $(3\lambda-3) \times (3\lambda-3)$ matrix $L_{\lambda-1}$ that has no λ times repeated entries in its rows and in its columns and does not contain a 2×2 submatrix of distinct elements.

For $\lambda \geq 3$ we show slightly more, that $n_\lambda(2, 2; m) \leq m - 2$ holds with $m = 3\lambda - 2$. This together with the $(m-1) \times (m-1)$ extremal matrix $L_{\lambda-1}$ will prove the claim that $n_\lambda(2, 2) = 3\lambda - 2$.

Assume that there is an $m \times (m-2)$ matrix $A = (a_{ij})$ with elements repeated less than λ times in rows and in columns and with no 2×2 submatrix of distinct elements.

Following the pigeonhole argument of the proof of Theorem 3 for $r = 2$, the formula (2) reduces to

$$\begin{aligned} t &= p(\lambda, m, 2)(m-2)/\tbinom{m}{2}. \\ &= \big(3(\lambda-1)^2 + 3(\lambda-1)\big)(3\lambda-4)/\tbinom{3\lambda-2}{2} \\ &= \tfrac{2\lambda(3\lambda-4)}{3\lambda-2} > 2\lambda - 2\ , \end{aligned}$$

where the last inequality holds since $\lambda \geq 3$. Hence there are at least $2\lambda - 1$ distinct pairs present in some pair of rows, w.l.o.g., in row 1 and 2 and in columns $1, 2, \ldots, 2\lambda - 1$.

Construct a graph G with vertex set equal to the set of entries of A and edges $a_{1i}a_{2i}$ for $i = 1, 2, \ldots, 2\lambda - 1$. Since A has no 2×2 submatrix of distinct elements, G has no pair of independent edges. Thus G is either a star or a triangle. Assume first that G is a star. Then the central vertex of the star corresponds to an entry of A which occurs in every pair $\{a_{1i}, a_{2i}\}$, $i = 1, 2, \ldots, 2\lambda - 1$. But then either row 1 or row 2 contains an element of A repeated at least λ times. Hence G is a triangle.

Suppose the three vertices of G correspond to entries 1, 2, 3. Then the ordered pairs (a_{1i}, a_{2i}) can be one of the six pairs $(1, 2)$, $(2, 3)$, $(3, 1)$, $(2, 1)$, $(3, 2)$, $(1, 3)$. If all these pairs occur then, after appropriate column permutations, one obtains that the top left corner looks like the matrix

$$T = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}\ .$$

Suppose now that at least one of those pairs is missing, say $(1, 3)$ does not occur. Each of the two rows a_{1i}, a_{2i}, $i = 1, 2, \ldots, 2\lambda - 1$ must contain all entries from $\{1, 2, 3\}$, so we deduce that the pairs $(1, 2)$ and $(2, 3)$ must occur. Also $(3, 1)$ must occur, since otherwise G would be a star. Hence w.l.o.g., the pairs $(1, 2)$, $(2, 3)$, $(3, 1)$ occur, and we may assume again that the top left 2×3 submatrix of A looks like T.

Since each column of A contains $3\lambda - 2$ elements, with repetition at most $\lambda - 1$, each column must contain at least four distinct entries. In particular the first column must contain an entry, say 4, not in $\{1, 2, 3\}$. W.l.o.g., the top left corner looks like

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 4 & a_{32} & a_{33} \end{pmatrix}$$

Since there is no rainbow 2×2 submatrix, $a_{32} \in \{2, 4\}$ and $a_{33} \in \{1, 4\}$. First assume $a_{32} = 4$. Then $a_{33} = 4$, and indeed all of the elements $a_{3i} = 4$ for $i = 1, 2, \ldots, 2\lambda - 1$ (check each of the six possible values for a_{1i}, a_{2i}). This gives too many 4's in this row. Thus $a_{32} = 2$, and similarly $a_{33} = 1$. We obtain the submatrix

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 4 & 2 & 1 \end{pmatrix}$$

Now the second column must also contain at least four distinct entries

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 4 & 2 & 1 \\ a_{41} & b & a_{43} \end{pmatrix}$$

where $a_{42} = b \geq 4$. First assume $a_{41} = b$. Then $a_{43} = b$, and indeed all of the elements $a_{4i} = b$ for $i = 1, 2, \ldots, 2\lambda - 1$ (check each of the six possible values for a_{1i}, a_{2i}). This gives too many b's in this row. Thus $a_{41} = 2$, and similarly $a_{43} = 3$.

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 4 & \mathbf{2} & \mathbf{1} \\ 2 & \mathbf{b} & \mathbf{3} \end{pmatrix}$$

But this gives a 2×2 submatrix of distinct entries on rows 3, 4 and columns 2, 3. $\square$

Before stating the next theorem, we prove a useful observation.

Lemma 6. *If a matrix A has no 2×2 submatrix with distinct entries, then either A has a column with all entries equal, or for any two columns, there is an entry that occurs in both columns.*

Proof. Pick two columns, and suppose no entry of the first occurs in the second. Construct a (bipartite) graph with vertex set equal to the entries and edges joining two entries x and y when there is a row with x in the first column and y in the second column. Since A has no rainbow 2×2 submatrix, G does not have two independent edges. Since G is bipartite, G must be a star $K_{1,r}$. But then one of the columns of A contains only one entry. $\square$

Theorem 7. *If $\lambda \geq 2$ then $n_\lambda(2, 2; \lambda) = \lambda^2$, and the extremal matrix is unique up to permutation of rows, columns, and entries.*

Proof. For $\lambda = 2$ it is easy to check that

$$C = \begin{pmatrix} 0 & 2 & 1 \\ 1 & 0 & 2 \end{pmatrix}$$

is the only extremal matrix up to isomorphism, so $\lambda \geq 3$ is assumed for the rest of the proof.

Let A be a $\lambda \times n$ matrix, $n \geq \lambda^2 - 1$, with no row or column containing an entry at least λ times and no 2×2 submatrix with distinct entries. Each column must have at least two distinct entries. Call a column a *2-column* if it has exactly two distinct entries. Construct a graph G with vertex set equal to the set of all entries in A that occur in some 2-column and edges xy whenever a 2-column exists with entries x and y. By Lemma 6, G

cannot contain two independent edges xy and zw. Hence G is either a K_3 or a star $K_{1,r}$. We shall show that G is a star.

Assume there are exactly $\mu(\lambda-1)$ 2-columns (where μ is not necessarily an integer). Now if a is a 2-column, then $p(a,2)\geq\lambda-1$, and if a is a non-2-column then $p(a,2)\geq 2\lambda-3$. Hence the counting argument of Theorem 3 gives

$$t\binom{\lambda}{2}=\mu(\lambda-1)^2+(n-\mu(\lambda-1))(2\lambda-3)\leq 3(\lambda-1)\binom{\lambda}{2}$$

Assuming $n\geq\lambda^2-1$ this gives (after simplifying) $\mu\geq\frac{\lambda+3}{2}\geq 3$. Now any entry can occur at most $\lambda(\lambda-1)$ times in A, so there are at most $|V(G)|\lambda(\lambda-1)$ entries in the 2-columns. If the 2-columns do not make up the whole of A then in fact we have strictly fewer entries, since otherwise every other column would have all entries not from $V(G)$, contradicting Lemma 6. Hence $\mu(\lambda-1)\lambda<|V(G)|\lambda(\lambda-1)$ and so $|V(G)|>\mu\geq 3$ (or $|V(G)|\geq\mu=n/(\lambda-1)\geq\lambda+1>3$ if all columns are 2-columns). Hence G is a star $K_{1,r}$ with $r\geq 3$.

Let the entry corresponding to the center of this star be 0. Since there can be at most $\lambda(\lambda-1)$ 0-entries in A, there are at least $\lambda-1$ columns which do not contain the entry 0, call these *r-columns.* Each r-column must contain each of the r entries corresponding to the leaves of the $K_{1,r}$ otherwise we could find a 2-column with entries distinct from all the entries of this r-column, contradicting Lemma 6. Since there are at most $\lambda(\lambda-1)$ 0-entries and $\mu(\lambda-1)>\frac{1}{2}\lambda(\lambda-1)$ 2-columns, each of which contains at least one 0 entry, there must be a 2-column with exactly one 0-entry, w.l.o.g., column 1 is $(0,1,1,\dots)^T$.

Suppose an r-column (which has no 0 entries) has entry $x\neq 1$ in row 1. If there is an entry $y\neq 1$ in row $i>1$ then $x=y$. Thus the only entries in this r-column are 1 and x, contradicting the fact that an r-column contains at least $r\geq 3$ distinct entries. Thus row 1 must contain the entry 1. Applying this argument to each r-column in turn shows that there can be at most $\lambda-1$ r-columns, otherwise row 1 would contain too many 1's. Hence $n=\lambda^2-1$ and there are exactly $\lambda-1$ r-columns and all the other $\lambda(\lambda-1)$ columns contain exactly one 0-entry.

Suppose a 2-column has 0 in the row i, and all other entries equal to $x\neq 0$. By the argument above, all $\lambda-1$ r-columns must have entry x in row i. Since there must be some row where some r-column does not contain x, there must be a row where every 2-column with entries $\{0,x\}$ must actually have the entry x. Thus there are at most $\lambda-1$ such 2-columns for each value of x. Hence $\mu(\lambda-1)\leq r(\lambda-1)$ and so $\mu\leq r$.

Each r-column, a say, contains (at least) r distinct values, so $p(a,2)\geq(r-1)(\lambda-r+1)+\binom{r-1}{2}=(r-1)(2\lambda-r)/2$. For any 2-column, $p(a,2)\geq\lambda-1$, and any column that is neither a 2-column nor an r-column, $p(a,2)\geq 2\lambda-3$. Hence using the counting argument of Theorem 3 we have

$$(\lambda-1)(r-1)(2\lambda-r)/2+\mu(\lambda-1)^2+(n-(\mu+1)(\lambda-1))(2\lambda-3)\leq 3(\lambda-1)\binom{\lambda}{2}.$$

Using $n=\lambda^2-1$ this simplifies to

$$(2\lambda-r)(r-1)+2\mu(\lambda-1)+2(\lambda-\mu)(2\lambda-3)\leq 3\lambda(\lambda-1),$$

and so

$$\begin{aligned}(2\lambda-r)(r-1)+2(\mu-\lambda)(\lambda-1)+2(\lambda-\mu)(2\lambda-3) &\le \lambda(\lambda-1),\\ (\lambda-r)(r-1)+(\lambda-\mu)(2\lambda-4) &\le \lambda(\lambda-1)-\lambda(r-1),\\ (\lambda-r)(r-1-\lambda)+(\lambda-\mu)(2\lambda-4) &\le 0, \qquad (4)\end{aligned}$$

which with $\mu \le r$ gives

$$(\lambda-r)(\lambda+r-5) \le 0.$$

Since $\lambda, r \ge 3$, we must have $r \ge \lambda$. But the r-columns have at least r distinct entries, so $r \le \lambda$. Hence $r = \lambda$, and so by (4), $\mu = \lambda$. Therefore, there are $\lambda(\lambda-1)$ 2-columns consisting of a single 0 and $\lambda-1$ copies of some $x \neq 0$, and the r-columns are $\lambda-1$ copies of the same column containing λ distinct entries. Furthermore, as we noticed above, in each 2-column with entry $x \neq 0$ the zero occurs in the row that contains x in the r-columns. The $\lambda \times (\lambda^2-1)$ extremal matrix is therefore unique up to permutation of rows, columns, and the values of the elements (see the matrix below where each entry corresponds to a $1 \times (\lambda-1)$ block of identical elements):

$$\begin{pmatrix} 0 & 2 & 3 & \dots & \lambda-1 & \lambda & 1 \\ 1 & 0 & 3 & \dots & \lambda-1 & \lambda & 2 \\ 1 & 2 & 0 & \dots & \lambda-1 & \lambda & 3 \\ \cdot & & & \cdot & & & \cdot \\ \cdot & & & & \cdot & & \cdot \\ \cdot & & & & & \cdot & \cdot \\ 1 & 2 & 3 & \dots & 0 & \lambda & \lambda-1 \\ 1 & 2 & 3 & \dots & \lambda-1 & 0 & \lambda \end{pmatrix}$$

□

4 Designs and extremal matrices

Theorem 8. $n_3(3,3;5) = n_3(3,3;6) = 71$.

Proof. Theorem 3 shows that 71 is an upper bound for both functions:

$$1 + \binom{5}{3} \times 7 \times 2 \times 2 \times p(3,5,3)^{-1} = 1 + \binom{6}{3} \times 7 \times 2 \times 2 \times p(3,6,3)^{-1} = 71.$$

To show that there are extremal 5×70 and 6×70 matrices, it is enough to construct the 6×70 extremal matrix for $m = 6$ since deleting any row from it gives an extremal matrix for $m = 5$.

For i an integer, let

$$F_i = \begin{pmatrix} p_i^1 & p_i^2 & \dots & p_i^7 \\ p_i^2 & p_i^3 & \dots & p_i^1 \\ p_i^4 & p_i^5 & \dots & p_i^3 \end{pmatrix}$$

be the line matrix of a Fano plane (projective plane of order 2) on the point set $\{p_i^1, \ldots, p_i^7\}$ listing the lines in its columns cyclically. Note that any two columns of F_i have a common entry. Let $F_1, \ldots, F_5$ be the line matrices of pairwise point disjoint Fano planes.

Consider a factorization $\Pi = \{M_1, \ldots, M_5\}$ of a clique K_6 into 1-factors on the vertex set $[6] = \{1, \ldots, 6\}$ (this is unique up to isomorphism, see [4]). Let $M_i = \{e_1^i, e_2^i, e_3^i\}$, and for each $i = 1, \ldots, 5$, define a 6×7 matrix F_i^* by "blowing up" F_i along M_i as follows. If $k \in e_j^i$ for some $j \in \{1, 2, 3\}$, then let the kth row of F_i^* be equal to the jth row of F_i.

We show that the 6×35 matrix A obtained as the (horizontal) concatenation of $F_1^*, \ldots, F_5^*$ has no 3×3 submatrix D with distinct entries. Assume on the contrary that such a D exists and let $a, b, c \in [6]$ be its row indices in A. Then ab, ac, and bc belong to distinct 1-factors in Π, say M_3, M_4, and M_5, respectively. To avoid identical entries in D the columns of D cannot be taken from the concatenation of F_3^*, F_4^*, and F_5^*. Thus the three columns must belong to the concatenation of F_1^* and F_2^* in A. Hence two columns must belong to the same 6×7 line matrix, say F_1^*. But any two columns of F_1, and thus F_1^*, have a common entry, a contradiction.

Duplicating all columns of A we get the required 6×70 matrix with the desired properties. $\square$

We conclude the paper by showing that the extremal matrix of Theorem 8 is not an isolated example, the upper bound in Theorem 3 is sharp if certain designs exist. We illustrate it only for $r = 3$. A 3-(v, k, λ_3) design is a collection of k-element blocks on a v-element ground set V such that each triplet of V is contained in precisely λ_3 blocks. We call a 3-design *pairable* if its blocks can be partitioned into point disjoint pairs.

Theorem 9. *Assume that for an odd integer $m \geq 7$ there exists a pairable 3-$(m, \frac{m-1}{2}, \lambda_3)$ design. Then the upper bound in Theorem 3 is sharp with parameters $\lambda = \frac{m+1}{2}$, $r = 3$, and $s = \frac{6\lambda_3(m-1)}{(m-3)(m-5)} + 1$.*

Proof. Set $\lambda = \frac{m+1}{2}$, and let $\mathcal{D}$ be the design specified in the theorem. Consider a partition of the b blocks of $\mathcal{D}$ into pairs of disjoint blocks (B_{2i-1}, B_{2i}), $i = 1, 2, \ldots, b/2$. We define $b/2$ partitions of V as follows: let $M_i = \{E_1^i, E_2^i, E_3^i\}$ where $E_1^i = B_{2i-1}$, $E_2^i = B_{2i}$, and $E_3^i = V \setminus (B_{2i-1} \cup B_{2i})$. Note that $|E_1^i| = |E_2^i| = \lambda - 1$ and $|E_3^i| = 1$. Blow up the 3×7 matrix F_i as described in the proof of Theorem 8 along the partition M_i. Thus we get the $m \times 7$ matrices F_i^*, $i = 1, \ldots, b/2$. Concatenating (horizontally) the matrices F_1^*, F_2^*, $\ldots$, $F_{b/2}^*$ with pairwise disjoint entries and repeating each column $\lambda - 1$ times gives the matrix A of size $m \times 7(\lambda - 1)b/2$.

It is obvious that the entries of A are repeated at most $\lambda - 1$ times in rows and in columns. Consider a submatrix B with three rows in A with distinct entries. We may assume by rearranging that B uses the rows 1, 2, and 3.

In the 3-design $\mathcal{D}$ there are $3\lambda_2 - 2\lambda_3$ blocks containing at least two elements of $\{1, 2, 3\} \subseteq V$, where λ_2 is the number of blocks covering a given pair of V. If a partition M_i, for some $1 \leq i \leq b/2$, contains such a block, then two rows of F_i^* among rows $1, 2$, and 3 are identical. Therefore, B has no column from the corresponding F_i^*. Because one partition may contain at most one block containing at least two elements of $\{1, 2, 3\}$, there are $t = b/2 - 3\lambda_2 + 2\lambda_3$ partitions M_j such that their blocks contain at most one element of $\{1, 2, 3\} \subseteq V$. Because any two columns of the line matrix of a Fano plane F_j have a common entry, B has at most one column from each of the t corresponding matrices F_j^*. Thus B cannot have $t + 1$ or more columns. Using that

$$b = \lambda_3 \tbinom{m}{3} / \tbinom{(m-1)/2}{3} = \tfrac{8\lambda_3 m(m-2)}{(m-3)(m-5)}, \qquad \text{and} \qquad \lambda_2 = \tfrac{2\lambda_3(m-2)}{(m-5)},$$

we obtain

$$t = b/2 - 3\lambda_2 + 2\lambda_3 = s - 1 \ .$$

Hence A has no $3 \times s$ submatrix with distinct entries. An easy calculation shows that the number of columns of A is one less than the upper bound in (3) of Corollary 4 with $\lambda = \frac{m+1}{2}$, $r = 3$, and $s = \frac{6\lambda_3(m-1)}{(m-3)(m-5)} + 1$. Therefore A is extremal and Theorem 3 is sharp as stated. □

Computer search shows that there is only one 3-$(17, 8, 14)$ design, the one due to A. Brouwer [3]: it is generated by the affine group GA17 acting on $\mathbb{F}_{17}$ with generator blocks $B_1 = \{1, 2, 3, 4, 5, 6, 7, 8\}$ and $B_2 = \{1, 2, 4, 8, 9, 13, 15, 16\}$ (the squares). This design is not pairable because the disjointness relation on the blocks has odd components (cycles of length 17). Using Füredi's theorem applied in the proof of Lemma 1, it is not difficult to see that this implies that the integral bound of Theorem 3 for the parameters $\lambda = 9$, $r = 3$, $s = 9$, $m = 17$ is not sharp. However, taking each block of 3-$(17, 8, 14)$ twice results in a pairable 3-$(17, 8, 28)$ design for which Theorem 9 is applicable. Therefore Theorem 3 is sharp for $\lambda = 9$, $r = 3$, $s = 17$, $m = 17$.

It is easy to see that Theorem 9 gives infinitely many examples for $r = 3$ with equality in Theorem 3. Probably t-designs can be used for $t > 3$ as well to find further cases when Theorem 3 is sharp.

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