Interference percolation

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Abstract

Let G be an infinite connected graph with minimum degree δ and maximum degree Δ. Let G_p be a random induced subgraph of G obtained by selecting each vertex of G independently with probability p, $0 < p < 1$, and let $G_p^{\le k}$ be the induced subgraph of G_p obtained by deleting all vertices of G_p with degree greater than k in G_p. We show that if $\delta \ge 6$ and Δ/δ is not too large then $G_p^{\le 3}$ almost surely has no infinite component. Moreover, this result is essentially best possible since there are examples where $G_p^{\le k}$ has an infinite component (a) when $\delta = \Delta = 3$, 4, or 5, and $k = 3$; (b) when $\Delta \gg \delta$ for any δ and $k = 3$; and (c) when $\delta = \Delta$ for any $\delta \ge 3$ and $k \ge 4$. In addition, we show that if G is the d-dimensional lattice $\mathbb{Z}^d$ then $G_{1/d}^{\le 4}$ almost surely has an infinite component for sufficiently large d.

1 Introduction

We consider the following problem. Suppose we have a communication network consisting of transceivers (vertices) and communication links (edges) forming a graph G. Suppose at some time each transceiver is active independently with probability p. We suppose that a message can be relayed through active transceivers, but with the restriction that

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if a transceiver has too many active neighbors, then interference between the signals means that this transceiver cannot effectively relay a message. We wish to know whether a message can be propagated effectively. More formally, let G be a countably infinite graph and let G_p be the random induced subgraph of G obtained by selecting each vertex of G independently with probability p. Let $G_p^{\leq k}$ be the induced subgraph of G_p obtained by deleting all vertices of G_p with degree greater than k. Define

$$P_k(G) = \{p \in (0,1) : \mathbb{P}(G_p^{\leq k} \text{ has an infinite component}) > 0\}.$$

If $p \in P_k(G)$ then we say that for site probability p we have *k-interference percolation on G*. Note that there is no monotonicity in p. Indeed, we expect percolation to fail in general both for very small p (when there is no percolation even without interference) and for very large p (when almost all vertices of G_p are deleted due to interference). We will be mainly interested in whether $P_k(G)$ is empty or not: we say G *k-percolates* if $P_k(G) \neq \emptyset$, i.e., for some p with $0 < p < 1$, $G_p^{\leq k}$ has an infinite component with positive probability. In fact, for *locally finite* graphs (graphs for which each vertex has finite degree), the event that $G_p^{\leq k}$ has an infinite component is unaffected by changing the state of a finite number of vertices. Thus, by the Kolmogorov 0–1 law, the probability that there is an infinite component is either 0 or 1. This may fail for graphs that are not locally finite. As an example, let G be a locally finite graph that k-percolates and form a new graph G' by adding $k+1$ new vertices to G, each joined to all the vertices of G. Let $p \in P_k(G)$. Then with probability p^{k+1}, all the additional vertices lie in G'_p, and so $G_p'^{\leq k}$ is almost surely empty. But with probability $(1-p)^{k+1}$, none of the additional vertices lie in G'_p: in this case $G_p'^{\leq k}$ almost surely has an infinite component. Thus

$$(1-p)^{k+1} \leq \mathbb{P}(G_p'^{\leq k} \text{ has an infinite component}) \leq 1 - p^{k+1},$$

i.e., the probability that with site probability p we have k-interference percolation on G is strictly between 0 and 1. From now on we shall only consider locally finite graphs.

For a graph G, write $\delta = \delta(G)$ and $\Delta = \Delta(G)$ for the minimum and maximum degrees of the vertices of G respectively. If the degrees of the vertices of G are unbounded then we set $\Delta(G) = \infty$. Our main result is the following.

Theorem 1. *Any graph G with $\delta(G) \geq 6$, $\Delta(G) < 2.372(\delta(G) - 3)$, and $(\delta, \Delta) \neq (6, 7)$ does not 3-percolate.*

We shall also show that this result is false if we do not bound the maximum degree.

Theorem 2. *For all δ, there exists a graph with minimum degree δ and finite maximum degree that 3-percolates.*

The maximum degree required for the proof of Theorem 2 is exponential in δ, while by Theorem 1 we know that Δ must be at least a linear multiple of δ if G is to 3-percolate. This suggests the following question.

Question 1. *What is the smallest value of Δ, as a function of δ, such that there exists a 3-percolating graph G with minimum and maximum degrees δ and Δ respectively.*

Although Theorem 2 shows that in general a bound on Δ is required in Theorem 1, there are some cases when Theorem 1 holds without an upper bound on Δ. One case is when G is a tree.

Theorem 3. *If T is a tree with minimum degree at least 6 then T does not 3-percolate.*

This might seem somewhat surprising since one generally expects percolation on trees to be easier than on general graphs with vertices of the same degree. However, cycles in G can sometimes make percolation easier when there is interference.

Showing that general graphs *do* k-percolate is somewhat harder. However, for trees we prove the following.

Theorem 4. *The infinite d-regular tree 3-percolates for $3 \leq d \leq 5$ and 4-percolates for all $d \geq 3$. Moreover, for any $k \geq 4$ and $\delta \geq 3$ there is a $\Delta = \Delta(k, \delta)$ such that any tree with minimum degree δ and maximum degree at most Δ k-percolates, but there exists a tree with minimum degree δ and maximum degree $\Delta + 1$ that does not k-percolate.*

The function $\Delta(k, \delta)$ can be calculated fairly easily. Some values are given in Table 1. It is worth noting that for *regular* k-percolating trees, the set $P_k(G)$ of p for which percolation occurs is an interval. However, this is not true in general, even for trees. Indeed, if G is obtained by joining two (locally finite) graphs G_1 and G_2 at a single vertex then $P_k(G) = P_k(G_1) \cup P_k(G_2)$. It is easy to find G_1 and G_2 for which the closures of $P_k(G_1)$ and $P_k(G_2)$ are disjoint, so that $P_k(G)$ is not an interval.

As another example we show that the (generalized) Gilbert model [3] does not 3-percolate.

Theorem 5. *Let G be an infinite graph with vertex set given by a uniform Poisson point process in $\mathbb{R}^d$ for some $d \geq 2$, and edges uv whenever $u - v$ lies in some fixed bounded symmetric subset $A \subseteq \mathbb{R}^d$. Then G does not 3-percolate.*

One would expect that this model would k-percolate for sufficiently large k and sufficiently large intensity of the Poisson process. Note that if the intensity of the Poisson process is too small, then no percolation occurs even with $p = 1$ and without removing high degree vertices. On the other hand, a Poisson process of intensity λ in which each vertex is kept with probability p is just a Poisson process of intensity $p\lambda$. Thus k-percolation with a given λ and p implies k-percolation with any $\lambda' \geq \lambda$ and $p' := p\lambda/\lambda'$. Hence k-percolation is independent of the intensity λ provided λ is sufficiently large.

In the special case when A is a disk in $\mathbb{R}^2$, numerical simulations suggest that G 7-percolates for large λ, but does not 6-percolate. It seems likely that this is the worst case.

Conjecture 1. *Let G be an infinite graph with vertex set given by a uniform Poisson point process in $\mathbb{R}^d$ of intensity λ for some $d \geq 2$, and edges uv whenever $u - v$ lies in some fixed bounded symmetric subset $A \subseteq \mathbb{R}^d$ of positive measure. Then G 7-percolates provided λ is sufficiently large.*

The intuition behind this conjecture is that as one departs from the 2-dimensional disk the graph becomes more tree-like. By the above mentioned independence on the intensity, it should therefore behave more like an almost regular tree of very high degree, which we know 4-percolates. Additional conjectures can be made in higher dimensions. For $d \geq 5$ it appears that even the case when A is a sphere gives 4-percolation, and for $d = 3, 4$, spheres give 5-percolation. Perhaps these are the worst cases in their respective dimensions.

In the case of a disk in $\mathbb{R}^2$ we can give a (rather crude) positive result.

Theorem 6. *Let G be an infinite graph with vertex set given by a uniform Poisson point process in $\mathbb{R}^2$, and edges uv whenever the Euclidean distance $\|u - v\| \leq R$, where R is any sufficiently large constant (depending on the intensity of the Poisson process). Then G 36-percolates.*

Finally, for lattices of high dimension we prove the following precise threshold. Here $\mathbb{Z}^d$ represents the lattice with vertex set $\mathbb{Z}^d$ and edges joining vertices which are at Euclidean distance 1.

Theorem 7. *For sufficiently large d the lattice $\mathbb{Z}^d$ 4-percolates but does not 3-percolate.*

Numerical simulations strongly suggest the following conjecture.

Conjecture 2. *For all $d \geq 2$ the lattice $\mathbb{Z}^d$ 4-percolates, but does not 3-percolate.*

For $d \geq 3$ absence of 3-percolation follows from Theorem 1, but this still leaves the question of 3-percolation open for $d = 2$. For $d = 2$, 4-percolation is clear as there are no vertices of degree more than 4 and site percolation occurs on $\mathbb{Z}^2$ for large p. However this still leaves open the question of the existence of 4-percolation for small $d > 2$.

Throughout this paper we shall use standard terminology from graph theory. In particular, $V(G)$ and $E(G)$ will denote the set of vertices, respectively edges, of the graph G. If $v \in V(G)$, $N_G(v)$ will denote the set of neighbors of v in G, and $\deg_G(v) = |N_G(v)|$ will denote the degree of the vertex v. Recall that a subgraph H of G is called *induced* if every edge of G whose endpoints lie in $V(H)$ is also an edge of H, and a set S of vertices is *independent* if there are no edges in G between vertices of S.

2 k-percolation on trees

In this section we deal with the simple case of percolation on trees. Although the results of this section are relatively straightforward, they provide a number of instructive examples and set the stage for later sections.

For all $k, d \geq 2$, define

$$f_{k,d}(p) = \sum_{i=1}^{k-1} \tbinom{d-1}{i} i p^i (1-p)^{d-1-i} \tag{1}$$

to be the expected value of a Binomial $\mathrm{Bin}(d-1, p)$ random variable that has been truncated so that values $\geq k$ are replaced with zero.

Lemma 8. *Let T be a locally finite tree and let $D = \{\deg_T(v) : v \in V(T)\}$ be the set of degrees of vertices in T. Fix $0 < p < 1$. If $f_{k,d}(p) > 1$ for all $d \in D$ then $T_p^{\leq k}$ almost surely has an infinite component. If $f_{k,d}(p) \leq 1$ for all $d \in D$ then $T_p^{\leq k}$ almost surely does not have an infinite component.*

Proof. Fix a root vertex v_0 of T and consider the subtree T' of T_p consisting of vertices v_n such that each of the vertices $v_1, \dots, v_{n-1}$ of the unique path $v_0 \dots v_n$ in T lies in T_p and has at most k neighbors in T_p. Note that in particular we do not require v_n (or v_0) to have at most k neighbors, so v_n may not lie in $T_p^{\leq k}$. However, the event $v_n \in V(T')$ depends only on the random process up to the level of v_n. Fixing $v_n \in V(T')$, $n > 0$, the number Y_{v_n} of children of v_n in T' is given by a Binomial distribution $\mathrm{Bin}(\deg_T(v_n)-1, p)$, except that if this gives more than $k-1$ vertices, then v_n has more than k neighbors in T_p (since v_{n-1} is also a neighbor) and so there are no children of v_n in T'. Hence the mean number of children $\mathbb{E}Y_{v_n}$ is given by $f_{k,\deg_T(v_n)}(p)$. Since the number of children of v_n is independent for different choices of v_n, T' is a 'Galton-Watson'-like tree. The only slight complication is that the distribution of the number of children of $v \in T'$ can vary depending on the degree of v in our original tree T.

Let $p_{n,v}$ be the probability of a path of length n existing in T' starting from v and going away from the root v_0. Then $p_{n+1,v} = \mathbb{E}(1 - \prod_i (1 - p_{n,v_i}))$ where the product is over all

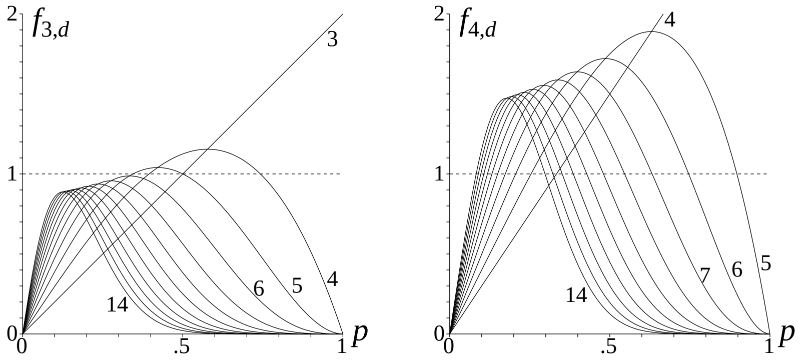

Figure 1: Functions $f_{k,d}$ for $k = 3, 4$ and $k \le d \le 14$.

children of v, and the expectation is over the random choice of children. For any p and k, $f_{k,d}(p) \to 0$ as $d \to \infty$, so under the assumption that $f_{k,d}(p) > 1$ for all $d \in D$, there can only be finitely many degrees $d \in D$. Then there are only a finite number of choices for the distribution of the random variable Y_v, and each is bounded (by k) with mean > 1. A simple calculation shows that there exists an $\alpha > 0$ such that $\mathbb{E}(1 - (1-\alpha)^{Y_v}) \ge \alpha$ for every possible distribution Y_v. Indeed, by convexity, $1 - (1-\alpha)^x \ge \frac{x}{k}(1 - (1-\alpha)^k)$ for $0 \le x \le k$, so

$$\mathbb{E}(1 - (1-\alpha)^{Y_v}) \ge \frac{1 - (1-\alpha)^k}{k\alpha}(\mathbb{E}Y_v)\alpha.$$

Now $\frac{1-(1-\alpha)^k}{k\alpha} \to 1$ as $\alpha \to 0$, so for sufficiently small α, $\frac{1-(1-\alpha)^k}{k\alpha}(\mathbb{E}Y_v) \ge 1$ for every v. Thus by induction on n, $p_{n,v} \ge \alpha$ for all n and v. Now by compactness, there exists an infinite path in T' (and hence an infinite path in $T_p^{\le k}$) with positive probability.

If $f_{k,d}(p) \le 1$ for all $d \in D$, then $f_{k,d}(p) = 1$ for only finitely many d and $f_{k,d}(p) \le 1-\varepsilon$ for all the rest. A simple calculation shows that for $\alpha \in (0, 1]$ there is a continuous function $f(\alpha) < \alpha$ such that $\mathbb{E}(1 - (1-\alpha)^{Y_v}) \le f(\alpha)$ for every choice of distribution Y_v. Indeed, $(1-\alpha)^n \ge 1 - n\alpha + \alpha^2$ for all $n \ge 2$, so $\mathbb{E}(1 - (1-\alpha)^{Y_v}) \le \alpha\mathbb{E}Y_v - \alpha^2\mathbb{P}(Y_v \ge 2)$. Thus we can take $f(\alpha) = \max\{(1-\varepsilon)\alpha, \alpha - c\alpha^2\}$ where c is the minimum value of $\mathbb{P}(Y_v \ge 2)$ over the finite set of distributions Y_v where $\mathbb{E}Y_v = f_{k,d}(p) = 1$. Note that none of these distributions give 1 with certainty, so $c > 0$. If we let $p_n = \sup_v p_{n,v}$, then $p_{n+1} \le f(p_n)$, and so $p_n \to 0$ as $n \to \infty$. Hence $T_p^{\le k}$ almost surely has no infinite component. $\square$

Before we continue, we note a few properties of the functions $f_{k,d}(p)$ (see Figure 1). An easy calculation shows that $\frac{d}{dp}\binom{n}{i}p^i q^{n-i} = \frac{n}{p}(\binom{n}{i}p^i q^{n-i} - \binom{n-1}{i}p^i q^{n-1-i})$, where $q = 1-p$. Thus from (1) we obtain

$$\tfrac{d}{dp} f_{k,d}(p) = \tfrac{d-1}{p}(f_{k,d}(p) - f_{k,d-1}(p)). \tag{2}$$

For $d = k$, $f_{k,d}(p) = (d-1)p$ is linear since it is just the expected value of a Binomial random variable $\mathrm{Bin}(d-1, p)$. For $d > k$, $f_{k,d}$ is a unimodal function. Indeed, if $f_{k,d}$ had a local minimum at $p_0 \in (0,1)$, say $f_{k,d}(p) = f_{k,d}(p_0) + (c + o(1))(p - p_0)^{2n}$, $c > 0$, then $f_{k,d-1}(p) = f_{k,d}(p_0) - (\frac{2npc}{d-1} + o(1))(p - p_0)^{2n-1}$, so $f_{k,d-1}$ would be decreasing near p_0. But by induction on d we may assume that $f_{k,d-1}$ is unimodal. Hence $f_{k,d-1}$ would be decreasing for all $p > p_0$, and so $f_{k,d}$ would be strictly increasing on $[p_0, 1]$, contradicting the fact that $f_{k,d}(1) = 0$. We also note that since the maximum value of $f_{k,d}$ occurs at a point where $f_{k,d} = f_{k,d-1}$, the maximum value of $f_{k,d}$ is a decreasing function of d.

Proof of Theorem 3. For $k = 3$, $f_{3,d}(p) = (d-1)pq^{d-2} + (d-1)(d-2)p^2q^{d-3}$ where $q = 1 - p$. It is easy to show that $f_{3,d}(p) < 1$ for all $d \geq 6$ and all $p \in (0,1)$. Indeed, by the above it is enough to prove this for the single function $f_{3,6}(p) = 5pq^4 + 20p^2q^3$ which has a maximum of $\frac{80}{81}$ at $p = \frac{1}{3}$. Hence Theorem 3 follows from Lemma 8. $\square$

Proof of Theorem 4. Now $f_{3,3}(p) = 2p > 1$ for $p \in (0.5, 1)$, $f_{3,4}(p) = 3p(1-p^2) > 1$ for $p \in [0.395, 0.742]$, and $f_{3,5}(p) = 4p(1 - 3p^2 + 2p^3) > 1$ for $p \in [0.345, 0.5)$ (see Figure 1). Lemma 8 then implies that the d-regular tree T_d 3-percolates for $d = 3$, 4, and 5. For $k = 4$ and large d let $p = \frac{2}{d}$. Then $q^d = (1 - \frac{2}{d})^d \to e^{-2}$, so

$$\begin{aligned} f_{4,d}(p) &= (d-1)pq^{d-2} + (d-1)(d-2)p^2q^{d-3} + \tfrac{1}{2}(d-1)(d-2)(d-3)p^3q^{d-4} \\ &\to (2 + 4 + 4)e^{-2} > 1 \qquad \text{as } d \to \infty. \end{aligned}$$

Since the maximum of $f_{4,d}$ is a decreasing function of d, we see that the maximum of $f_{4,d}$ must be greater than 1 for all $d \geq 3$. Hence by Lemma 8, T_d 4-percolates for all $d \geq 3$.

For fixed $d \geq k \geq 4$, $f_{k,d}(p) > 1$ holds for all p in some interval (p_-, p_+) (or $(p_-, 1]$ if $d = k$). Both $p_- = p_-(k,d)$ and $p_+ = p_+(k,d)$ are monotonically decreasing to 0 as a function of d. Hence if $f_{k,\delta}(p) > 1$ and $f_{k,\Delta}(p) > 1$ then $f_{k,d}(p) > 1$ for all $\delta \leq d \leq \Delta$. By Lemma 8, any tree with degrees between δ and Δ will k-percolate. Thus we can set $\Delta(k,\delta)$ to be the maximum Δ such that $p_+(k,\Delta) > p_-(k,\delta)$. For the converse, construct a tree level by level with branching ratio $\Delta = \Delta(k,\delta)$ on the levels $(2t)!$ to $(2t+1)! - 1$ and ratio $\delta - 1$ on levels $(2t+1)!$ to $(2t+2)! - 1$. For any $p \in (0,1)$, either $\alpha := f_{k,\delta}(p) \leq 1$ or $\beta := f_{k,\Delta+1}(p) \leq 1$. If $\min\{\alpha, \beta\} = \alpha < 1$ then the expected number of paths to level $(2t+2)!$ is at most $\alpha^{(2t+2)!}(\beta/\alpha)^{(2t+1)!} = (\alpha^{2t+1}\beta)^{(2t+1)!}$ which tends to 0 as $t \to \infty$. Similarly, if $\min\{\alpha, \beta\} = \beta < 1$ then the expected number of paths to level $(2t+1)!$ is at most $\beta^{(2t+1)!}(\alpha/\beta)^{(2t)!} = (\beta^{2t}\alpha)^{(2t)!}$ which tends to 0 as $t \to \infty$. In either case, the probability of a path of length n from the root vertex tends to 0 as $n \to \infty$, so there is no percolation. If $\alpha = 1 < \beta$ or $\alpha > 1 = \beta$, then by continuity there exists a p with $\alpha, \beta > 1$, so $p_+(k, \Delta+1) > p_-(k,\delta)$, contradicting the choice of Δ. Hence the only remaining case is when $\alpha = \beta = 1$, in which case the tree does not k-percolate by Lemma 8. $\square$

Table 1: Values of $\Delta(k,\delta)$ for small k and δ.

$k\backslash\delta$	3	4	5	6	7	8	9	10	100	1000	λ_+/λ_-
3	4	5	5	–	–	–	–	–	–	–	–
4	8	12	16	19	23	26	30	33	346	3466	3.4673050219
5	11	17	23	29	34	40	46	51	560	5643	5.6477466571
6	14	22	29	37	44	52	59	67	737	7442	7.4492851781
7	17	26	35	45	54	63	72	81	899	9082	9.0915646255
8	20	31	41	52	63	73	84	95	1054	10645	10.6565558375
9	23	35	47	59	72	84	96	108	1204	12161	12.1748285628
10	25	39	53	67	80	94	108	121	1351	13644	13.6592985801

Table 2: Values of $\lambda_\pm$ for small k and asymptotic values for large k.

k	λ_-	λ_+
4	1.1141416270	3.8630688584
5	1.0207039085	5.7646770872
6	1.0037312314	7.4770801849
7	1.0005963707	9.0969865684
8	1.0000832907	10.6574434289
9	1.0000102500	12.1749533554
10	1.0000011252	13.6593139497
$\to\infty$	$1+\frac{e^{-1}+o(1)}{(k-1)!}$	$k+\sqrt{(2+o(1))k\log k}$

Table 1 lists $\Delta(k,\delta)$ for small k and δ. One can show that $\delta p_\pm \to \lambda_\pm$ as $\delta\to\infty$, where $\lambda_\pm$ are the two roots of the equation

$$\sum_{i=1}^{k-1} e^{-\lambda}\frac{\lambda^i}{(i-1)!} = 1.$$

Hence the interesting range of p is when p is of order $\frac{1}{d}$. The constants $\lambda_\pm$ for small k and asymptotic values are given in Table 2. Since for large d, $p_\pm \sim \lambda_\pm/d$, $\Delta(k,\delta)/\delta \to \lambda_+/\lambda_-$ as $\delta\to\infty$.

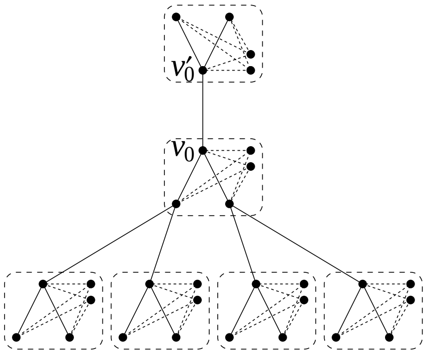

Figure 2: Construction used in the proof of Theorem 2 with $R = 2$.

3 3-percolating graphs with large minimum degree

For trees, $\delta \geq 6$ implies that G does not 3-percolate since for any $p \in (0, 1)$, $f_{3,d}(p) < 1$ when $d \geq 6$. However, it is possible for an arbitrary graph with large minimum degree to 3-percolate.

Proof of Theorem 2. We may assume $\delta > 3$ since otherwise we can just add a vertex of degree δ to the 3-regular infinite tree T_3. Fix $p = \frac{3}{4}$ and let R be large. Construct G as follows. Take T_3 and partition the vertices into subtrees T_i isomorphic to the first R levels of a binary tree (with $2^R - 1$ vertices). For example, take any edge v_0v_0' of T_3 and consider T_3 as two back-to-back infinite binary trees T and T' rooted at v_0 and v_0' respectively. In each of these binary trees take any vertex v that is at distance a multiple of R from the root and consider it as the root of a subtree T_i consisting of v and all its descendants for the next $R - 1$ levels. Then both T and T' are decomposed into trees T_i as required. For each i add $\delta - 3$ new vertices, and join them to every vertex of T_i (see Figure 2). The minimum degree is now δ and the maximum degree is $\Delta = 2^R - 1$ (provided of course that $2^R - 1 \geq \delta$). By amalgamating each T_i into a single vertex, T and T' can be thought of as trees of trees with "vertices" T_i, each of which has 2^R children. We count the number of paths from the root of T_i to the roots of its child trees T_j on the assumption that the $\delta - 3$ additional vertices do not lie in G_p. The total number of paths is 2^R and each is open (except possibly for the last vertex) with probability p^R. Thus the mean number of open paths is $p^R 2^R = 1.5^R$. The mean number of paths across each T_i to its children is then at least $0.25^{\delta-3}1.5^R$ which is > 1 if $R \geq 4\delta$. By comparison with a Galton-Watson process, regarding the T_i as nodes, we almost surely have an infinite component. □

Note that our example has a maximum degree of about 16^δ which, as mentioned in the introduction, is exponential in δ.

4 Absence of 3-percolation

To prove Theorem 1 we shall use a path-counting technique, showing that the expected number of induced paths of length n tends to zero as $n \to \infty$. The main complication is that one can encounter situations where, conditioned on the path getting to a certain vertex v, the number of possible extensions of this path to one new vertex is less than $\delta - 1$. This is due to the fact that, unlike for trees, many of the neighbors of v may be adjacent to earlier vertices on the path, and so are ineligible for use in extending the path. Moreover, conditioned on the path existing in $G_p^{\le k}$, these neighbors are very likely to be closed, and so do count to the degree of v in G_p. Thus we may be in a situation that locally looks like a k-regular tree for $k \in \{3, 4, 5\}$, and we know that these *do* 3-percolate. However these situations cannot be too frequent as then some previous vertex in the path would have too many neighbors. To deal with these issues we shall introduce certain weighted counts of paths, and prove bounds inductively on these.

Proof of Theorem 1. Fix some vertex v_0 and, for $n > 0$, let $\mathcal{P}_n$ be the set of all *induced* paths $P = (v_0, v_1, \ldots, v_n)$ of G of edge length n starting at v_0 (so the only vertices of P that are adjacent to v_i in G are $v_{i\pm1}$). Let $\mathcal{P}'_n$ be the (random) set of paths $(v_0, \ldots, v_n)$ of $\mathcal{P}_n$ that are subgraphs of G_p and for which the degree in G_p of $v_1, \ldots, v_{n-1}$ are all at most 3. Clearly, if the probability of an infinite component existing in $G_p^{\le 3}$ is positive, then for some v_0 there must be a positive probability that there is an infinite component containing v_0. Hence there is an infinite induced path in $G_p^{\le 3}$ starting at v_0 with probability at least some $\varepsilon_0 > 0$. Thus, in particular, $\mathbb{E}|\mathcal{P}'_n| \ge \varepsilon_0$ for all $n > 0$. We shall show that in fact $\mathbb{E}|\mathcal{P}'_n| < \varepsilon_0$ for sufficiently large n, so G does not 3-percolate.

Fix a path $P = (v_0, \ldots, v_n) \in \mathcal{P}_n$. For $i = 0, \ldots, n-1$, let $d_i = d_i(P)$ be the number of vertices adjacent to v_i that are not adjacent to $v_0, \ldots, v_{i-1}$. Thus d_i is the number of possible choices for v_{i+1} given $v_0, \ldots, v_i$. First we claim that

$$\sum_{P \in \mathcal{P}_n} \prod_{i=0}^{n-1} d_i^{-1} = 1. \tag{3}$$

Indeed, there are d_{n-1} paths $(v_0, \ldots, v_n) \in \mathcal{P}_n$ for each path $(v_0, \ldots, v_{n-1}) \in \mathcal{P}_{n-1}$, so $\sum_{P \in \mathcal{P}_n} \prod_{i=0}^{n-1} d_i^{-1} = \sum_{P' \in \mathcal{P}_{n-1}} d_{n-1} \prod_{i=0}^{n-1} d_i^{-1} = \sum_{P' \in \mathcal{P}_{n-1}} \prod_{i=0}^{n-2} d_i^{-1} = 1$ by induction on n. The case $n = 1$ of (3) is of course trivial.

Consider the event that $P \in \mathcal{P}'_n$. Let N_n be the set of neighbors of $\{v_1, \ldots, v_{n-1}\}$ in G that do not lie in P. Then $P \in \mathcal{P}'_n$ iff $V(P) \subseteq V(G_p)$ and for all $i = 1, \ldots, n-1$ there

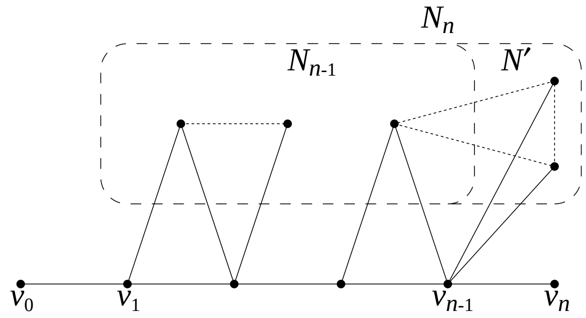

Figure 3: Path P and auxiliary graph H_n on vertex set N_n. The vertex v_{n-1} has $s = 1$ 'old' neighbors in N_{n-1} and $r = 2$ 'new' neighbors in $N' = N_n \setminus N_{n-1}$. Edges exist in H_n between vertices with common neighbors in $\{v_1, \ldots, v_{n-1}\}$.

is at most one vertex of $N_G(v_i) \cap N_n$ lying in G_p. (Since P is induced, each of these v_i is adjacent to precisely two vertices of P.) Construct an auxiliary graph H_n with vertex set N_n and edges joining two vertices u and v if and only if u and v have a common neighbor in $\{v_1, \ldots, v_{n-1}\}$ (see Figure 3). Thus for any $P \in \mathcal{P}_n$ we have $P \in \mathcal{P}'_n$ if and only if $V(P) \subseteq V(G_p)$ and $V(G_p) \cap N_n$ is an independent set in the graph H_n. Hence

$$\mathbb{P}(P \in \mathcal{P}'_n) = p^{n+1} \sum_{I \text{ independent}} \mathbb{P}(V(G_p) \cap N_n = I) = p^{n+1} \sum_I \prod_{v \in N_n} z(I, v), \tag{4}$$

where

$$z(I, v) = \begin{cases} 1 - p, & \text{if } v \notin I; \\ p, & \text{if } v \in I; \end{cases}$$

and the sum runs over all independent sets I of H_n.

We aim to bound this expression by induction on n, but to do this we shall need to consider a slight generalization. Fix a weight function $w \colon N_n \to \mathbb{Z}$ such that for all $v \in N_n$, $0 \le w(v) \le \Delta - 1$, and fix constants $\alpha \ge 1$ and $\beta \le \alpha$, (to be determined). Define the *partition function*

$$Z_{n,w} = \sum_I \prod_{v \in N_n} z_w(I, v),$$

where

$$z_w(I, v) = \begin{cases} q\alpha^{w(v)}, & \text{if } v \notin I; \\ p\beta^{w(v)}, & \text{if } v \in I; \end{cases}$$

and $q = 1 - p$.

Let $r = d_{n-1} - 1$ and $s = \deg_G(v_{n-1}) - 2 - r$. Then $|N_G(v_{n-1}) \cap N_n| = r + s$ with $r = |N_n \setminus N_{n-1}|$ 'new' neighbors and $s = |N_G(v_{n-1}) \cap N_{n-1}|$ 'old' neighbors. Define $w' \colon N_{n-1} \to \mathbb{Z}$ by

$$w'(v) = \begin{cases} w(v) + 1, & \text{if } v \in N_G(v_{n-1}); \\ w(v), & \text{if } v \notin N_G(v_{n-1}). \end{cases}$$

We shall compare $Z_{n,w}$ with $Z_{n-1,w'}$. Fix an independent set I_{n-1} of H_{n-1}. There are two cases. Assume first that $I_{n-1} \cap N_G(v_{n-1}) = \emptyset$. Then any extension of I_{n-1} to an independent set I_n of N_n can contain at most one vertex from $N' := N_n \setminus N_{n-1}$. Thus

$$\begin{aligned}
\sum_{\substack{I_n\colon \\ I_n \cap N_{n-1} = I_{n-1}}} \prod_{v \in N_n} z_w(I_n, v) &\le \prod_{v \in N_n} z_w(I_{n-1}, v) + \sum_{u \in N'} \prod_{v \in N_n} z_w(I_{n-1} \cup \{u\}, v) \\
&= \left(\prod_{v \in N'} q\alpha^{w(v)} \right) \left(1 + \sum_{u \in N'} \frac{p\beta^{w(u)}}{q\alpha^{w(u)}} \right) \prod_{v \in N_{n-1}} z_w(I_{n-1}, v) \\
&= q^r \alpha^{\sum_{v \in N'} w(v)} \left(1 + \sum_{u \in N'} \frac{p\beta^{w(u)}}{q\alpha^{w(u)}} \right) \alpha^{-s} \prod_{v \in N_{n-1}} z_{w'}(I_{n-1}, v).
\end{aligned}$$

(This is an inequality as I_{n-1} may not be independent in H_n and so the sum on the left hand side may be empty. In the last line we are using the fact $w'(v) = w(v) + 1$ only when $v \in N_G(v_{n-1})$, but then $v \notin I_{n-1}$.) On the other hand, if $I_{n-1} \cap N_G(v_{n-1}) \ne \emptyset$ then the only possible extension is $I_n = I_{n-1}$. Also $|I_{n-1} \cap N_G(v_{n-1})| = 1$, so

$$\prod_{v \in N_n} z_w(I_n, v) = q^r \alpha^{\sum_{v \in N'} w(v)} \alpha^{1-s} \beta^{-1} \prod_{v \in N_{n-1}} z_{w'}(I_{n-1}, v).$$

Now assume that

$$p(r+1) q^r \alpha^{\sum_{v \in N'} w(v)} \Big(1 + \sum_{v \in N'} \big(p\beta^{w(v)} / q\alpha^{w(v)} \big) \Big) \le (1 - \varepsilon) \alpha^s \tag{5}$$

and

$$p(r+1) q^r \alpha^{\sum_{v \in N'} w(v)} \le (1 - \varepsilon) \alpha^{s-1} \beta. \tag{6}$$

Then in both cases above,

$$p(r+1) \sum_{\substack{I_n\colon \\ I_n \cap N_{n-1} = I_{n-1}}} \prod_{v \in N_n} z_w(I_n, v) \le (1 - \varepsilon) \prod_{v \in N_{n-1}} z_w(I_{n-1}, v)$$

so that

$$\begin{aligned}
p d_{n-1} Z_{n,w} &= p(r+1) \sum_{I_n} \prod_{v \in N_n} z_w(I_n, v) \\
&\le (1 - \varepsilon) \sum_{I_{n-1}} \prod_{v \in N_{n-1}} z_{w'}(I_{n-1}, v) = (1 - \varepsilon) Z_{n-1,w'}.
\end{aligned}$$

Now fix n and $w = w_n = 0$. By induction

$$p^{n-1}\Big(\prod_{i=1}^{n-1} d_i\Big) Z_n \leq (1-\varepsilon)^{n-1} Z_{1,w_1},$$

where each $w_i = (w_{i+1})'$ used in the induction satisfies $0 \leq w_i(v) \leq \deg_G(v) - 1 \leq \Delta - 1$. Indeed, $w_i(v)$ is just the number of neighbors of v in $\{v_i, v_{i+1}, \ldots, v_{n-1}\}$. But $N_1 = \emptyset$ and so $Z_{1,w_1} = 1$. Thus

$$\begin{aligned}
\mathbb{E}|\mathcal{P}'_n| &= \sum_{P\in\mathcal{P}_n} \mathbb{P}(P \in \mathcal{P}'_n) = p^{n+1} \sum_{P\in\mathcal{P}_n} Z_n \\
&\leq p^2 d_0 (1-\varepsilon)^{n-1} \sum_{P\in\mathcal{P}_n} \Big(\prod_{i=0}^{n-1} d_i^{-1}\Big) Z_{1,w_1} \\
&\leq p^2 \Delta (1-\varepsilon)^{n-1}, \qquad (7)
\end{aligned}$$

where in the last step we have used (3). Thus for sufficiently large n, $\mathbb{E}|\mathcal{P}'_n| < \varepsilon_0$ contradicting our assumption that $\mathbb{E}|\mathcal{P}'_n| \geq \varepsilon_0$.

It remains to find $\alpha \geq 1$ and $\beta \leq \alpha$ satisfying (5) and (6) for all weight functions w with $0 \leq w(v) \leq \Delta - 1$. Let

$$A = \alpha^{\Delta-1}, \qquad B = \beta^{\Delta-1}. \qquad (8)$$

By convexity of α^i and β^j, (5) and (6) hold if they hold for all choices of weight function with $w(v) \in \{0, \Delta-1\}$. Setting i to be the number of $v \in N'$ with $w(v) = \Delta - 1$, it is enough if for all $0 \leq i \leq r$, $s \geq 0$, and $\delta - 2 \leq r + s \leq \Delta - 2$,

$$\begin{aligned}
p(r+1)q^{r-1}A^i(q + (r-i)p + ipB/A)) &\leq (1-\varepsilon)\alpha^s & (9)\\
p(r+1)q^r A^i &\leq (1-\varepsilon)\alpha^{s-1}\beta. & (10)
\end{aligned}$$

Since there are only a finite number of inequalities, it is enough if

$$\begin{aligned}
p(r+1)q^{r-1}A^i(q + (r-i)p + ipB/A)) &< \alpha^{\max\{\delta-r-2,0\}} & (11)\\
p(r+1)q^r A^r &< \alpha^{\max\{\delta-r-2,0\}-1}\beta. & (12)
\end{aligned}$$

where we have used $\alpha \geq 1$ and substituted the smallest possible value of s and, in (12), the largest possible value of i. The result now follows from the following lemma. □

Lemma 9. *For all $6 \leq \delta \leq \Delta \leq 2.372(\delta - 3)$, $(\delta, \Delta) \neq (6, 7)$, and for all $p \in (0, 1)$, there exist $\alpha \geq 1$ and $\beta \leq \alpha$ satisfying* (11) *and* (12) *for all $0 \leq i \leq r \leq \Delta - 2$, where A and B are defined as in* (8).

Proof. First consider (11) with $i = 0$.

$$f_{3,r+2} = p(r+1)q^{r-1}(q+rp) < \alpha^{\max\{\delta-r-2,0\}}, \tag{13}$$

where $f_{k,d}$ is as in Section 2. For $r \notin \{1,2,3\}$ the left hand side is < 1 for all p. If the left hand side is also < 1 for $r \in \{1,2,3\}$ then we can take $\alpha = \beta = 1$ and then all the inequalities (11) and (12) hold. Now assume the left hand side is ≥ 1 for some $r \in \{1,2,3\}$. We shall make the following assumptions.

$$\alpha(qA + pB) \leq \tfrac{2}{3}, \qquad \beta = \tfrac{2}{3}. \tag{14}$$

We shall show that all the inequalities (11) and (12) follow from (13) and (14). For $r \geq 2$, $i \geq 1$, inequality (11) with parameters (r,i) follows from the $(r-1,i-1)$ case of (11). Indeed,

$$\begin{aligned}
&p(r+1)q^{r-1}A^i(q+(r-i)p+ipB/A)) \\
&\qquad \leq p(r+1)q^{r-1}A^i(q+(r-i)p+(i-1)pB/A))(1+pB/qA) \\
&\qquad \leq \tfrac{r+1}{r} qA\alpha^{\max\{\delta-r-1,0\}}(1+pB/qA) && \text{from } (r-1,i-1) \text{ case} \\
&\qquad \leq \tfrac{3}{2}(qA+pB)\alpha^{\max\{\delta-r-1,0\}} && \text{since } r \geq 2 \\
&\qquad \leq \alpha^{\max\{\delta-r-2,0\}} && \text{by (14) and } \alpha \geq 1
\end{aligned}$$

For $r \geq 2$ the inequalities (12) follow the $r-1$ case of (12):

$$p(r+1)q^rA^r \leq (\tfrac{3}{2}qA)(prq^{r-1}A^{r-1}) < \alpha^{-1}\alpha^{\max\{\delta-r-1,0\}-1}\beta \leq \alpha^{\max\{\delta-r-2,0\}-1}\beta.$$

The $i = 0$ case of (11) is of course precisely (13), and the remaining inequalities follow from the $r = 1$ case of (13), i.e, $2p < \alpha^{\delta-3}$ (note $\delta \geq 6$). For the $(r,i) = (1,1)$ case of (11)

$$2pA(q+pB/A)) \leq 2p(qA+pB) \leq \tfrac{2}{3\alpha}2p < 2p < \alpha^{\delta-3},$$

and for the $r = 1$ and $r = 0$ cases of (12)

$$2pqA \leq (\tfrac{2}{3\alpha})2p < \tfrac{\beta}{\alpha}\alpha^{\delta-3} = \alpha^{\delta-4}\beta, \qquad \text{and} \qquad p < (\tfrac{2}{3})2p < \alpha^{\delta-3}\beta.$$

Hence we are done provided (13) and (14) both hold. By setting α to be slightly larger than the maximum of $f_{3,k}^{1/(\delta-k)}$, $k = r+2 \in \{3,4,5\}$, (so that (13) holds) it is then enough if

$$qf_{3,k}^{\Delta/(\delta-k)} < \tfrac{2}{3} - 2(\tfrac{2}{3})^{\Delta-1} \tag{15}$$

whenever $f_{3,k} \geq 1$ (using the trivial bound $\alpha p \leq \alpha \leq f_{3,k} \leq 2$).

For $\Delta \geq \delta \geq 60$, say, (15) holds when $qf_{3,k}^{\Delta/(\delta-k)} < 0.6666$. These inequalities impose a restriction on Δ only via the ratio $\Delta/(\delta-k)$, and by maximizing the left hand side over $p \in (0,1)$ it is enough that

$$\Delta \leq \min\big\{2.372(\delta-3), 2.415(\delta-4), 2.569(\delta-5)\big\} = 2.372(\delta-3).$$

For $\delta < 60$, inequalities (11) and (12) were checked directly by computer, choosing α and β as above. To deal with the fact that there are infinitely many values of p, we divided $[0, 1]$ into small ranges $[p_k, p_{k+1}]$ and checked the inequalities (11) and (12) for all p in this range by setting $p = p_{k+1}$ and $q = 1 - p_k$ so as to bound the left hand sides of (11) and (12) for all $p \in [p_k, p_{k+1}]$. All cases for which $\delta \leq \Delta \leq 2.372(\delta - 3)$ and $60 > \delta \geq 6$ were successful, except for the case $(\delta, \Delta) = (6, 7)$, which fails for, e.g., $p = 0.5$. $\square$

We finish this section with a useful Lemma.

Lemma 10. *For sufficiently large δ and Δ/δ sufficiently close to 1, the probability of an induced path of $G_p^{\leq 3}$ of edge length n starting at any given point of G is at most $p^2\Delta(0.842)^{n-1}$.*

Proof. Fix a point v_0 of G. Then in the proof of Theorem 1 we have by (7)

$$\mathbb{P}(\exists \text{ an induced path of length } n \text{ starting at } v_0 \text{ in } G_p^{\leq k}) \leq \mathbb{E}|\mathcal{P}'_n| \leq p^2\Delta(0.842)^{n-1},$$

provided that we can take $1 - \varepsilon = 0.842$ in (9) and (10). Thus it is enough to find $\alpha \geq 1$ and $\beta \leq \alpha$, depending only on p, such that for all $0 \leq i \leq r$,

$$p(r+1)q^{r-1}A^i(q + (r-i)p + ipB/A)) \leq c\alpha^{\max\{\delta-r-2,0\}}, \tag{16}$$

$$p(r+1)q^rA^r \leq c\alpha^{\max\{\delta-r-2,0\}-1}\beta, \tag{17}$$

where $c = 0.842$. Comparing (17) with the case $i = r$ of (16), we see that if we reduce c slightly to $c_1 < c$ in (16) we can take $\beta = (c_1/c)\alpha$ so that (17) follows. Then $B/A = (c_1/c)^{\Delta-1}$ tends to 0 as $\Delta \to \infty$. Thus for any fixed $\gamma < 1$ it is enough that

$$p(r+1)q^{r-1}A^i(q + (r-\gamma i)p) \leq c_1\alpha^{\max\{\delta-r-2,0\}}. \tag{18}$$

First we deal with the case when p is very small. In this case take $\alpha = 1$. The left hand side of (18) is maximized when $i = 0$, so we are reduced to checking the inequality $p(r+1)q^{r-1}(q+rp) \leq c_1$. Let $x = (r-1)p$. Then $p(r+1)q^{r-1}(q+rp) \leq (2p+x)e^{-x}(1+x) \leq 2p + x(x+1)e^{-x}$. This last expression is maximized when $x = (\sqrt{5}+1)/2$ and is less than $2p + 0.840$. Since c_1 can be chosen arbitrarily close to 0.842, the result follows for all $p < 0.001$. For other p we shall choose α so that $1 \leq A = \alpha^{\Delta-1} \leq 1 + p + p^2$, so qA is bounded away from 1. Then $p(r+1)q^{r-1}A^i(q + (r-\gamma i)p) \leq 3(r+1)^2(qA)^{r-1}$ tends rapidly to 0 as r grows. Thus (18) automatically holds for all large r. Since r and A are bounded and δ/Δ is close to 1, the right hand side is close to c_1A. Thus we are reduced to finding A with

$$p(r+1)q^{r-1}A^i(q + (r-\gamma i)p) \leq c_2A, \tag{19}$$

where we have reduced c a bit further to $c_2 < c_1$. This inequality is independent of A when $i = 1$, and it is this case that puts the most stringent bounds on c_2 (and hence c).

Maximizing $p(r+1)q^{r-1}(q+(r-0.99999)p)$ over all r and p gives a value just under 0.8419 with $r = 16$. Thus we take $c_2 = 0.8419$. To satisfy (19) when $i = 0$ we set $A = c_2^{-1} \sup_r p(r+1)q^{r-1}(q+rp)$. (If this is less than 1 then we can take $A = 1$ and then (19) holds for all i.) It now remains to check that $A \le 1+p+p^2$ and (19) also holds for all $i \ge 2$. We checked this by computer. To deal with the fact that there are infinitely many values of p, we once again divided $[0,1]$ into small ranges $[p_k, p_{k+1}]$ and checked (19) for all p in this range by setting $p = p_{k+1}$ and $q = 1-p_k$, so as to bound the left hand side of (19) for all $p \in [p_k, p_{k+1}]$. The value of A is chosen to be that for $p = p_{k+1}$, since it can be shown that A is an increasing function of p. Only a finite number of pairs (r,i) need to be checked as (19) automatically holds for large r since $qA < 1$. □

5 Continuum Percolation

Proof of Theorem 5. Let the vertices of the graph $G = G_A(\lambda)$ be given by a uniform Poisson process in $\mathbb{R}^d$ of intensity λ, and let uv be an edge of G whenever $u - v$ lies in a fixed symmetric bounded set $A \subseteq \mathbb{R}^d$ of positive measure $|A|$. As selecting points independently with probability p from a Poisson process of intensity λ/p yields a Poisson process of intensity λ, we can regard G as G'_p where $G' = G_A(\lambda/p)$. Let R be a bound on the length of an edge, so for example we can take $R = \operatorname{ess\,sup}_{x\in A} \|x\|$. Pick $\varepsilon > 0$. Let $t > 0$ be chosen so that $0.842^t < \varepsilon$. The number of neighbors N of a vertex of G' is given by a Poisson distribution with mean and variance $\mu = |A|\lambda/p$. Now by the Chernoff bound

$$\mathbb{P}(|N-\mu| \ge \varepsilon\mu) \le 2e^{-\varepsilon^2\mu/3}$$

for all $\varepsilon \le 1$. Thus for fixed $\varepsilon \le 1$, the probability that the degree of a vertex of G' is outside the range from $(1-\varepsilon)|A|\lambda/p$ to $(1+\varepsilon)|A|\lambda/p$ is bounded by $2e^{-\varepsilon^2|A|\lambda/(3p)}$. The expected number of vertices within $tR+1$ of the origin is at most $(2(tR+1))^d\lambda/p$. Moreover, as the vertices are distributed according to a Poisson process, conditioning on the presence of a vertex at a point does not affect its degree distribution. Thus by linearity of expectation, the expected number of vertices within $tR+1$ of the origin with exceptional degrees is bounded by $2e^{-\varepsilon^2|A|\lambda/(3p)}(2(tR+1))^d\lambda/p$ which tends rapidly to zero as $p \to 0$. Thus we can choose $p > 0$ sufficiently small so that with probability at least $1-\varepsilon$, all points of G' within $tR+1$ of the origin have degree between $(1-\varepsilon)|A|\lambda/p$ and $\Delta := (1+\varepsilon)|A|\lambda/p$. Let B be the unit ball about the origin, and note that there are on average $|B|\lambda/p$ vertices in B. By Lemma 10 applied to the part of the graph G' within distance $tR+1$ of the origin, and assuming ε is sufficiently small, the probability of a path of length t originating from some point in B is at most $(|B|\lambda/p)p^2\Delta\, 0.842^{t-1} = (1+\varepsilon)|A||B|\lambda^2\, 0.842^{t-1}$. Once again we have used linearity of expectation and the fact that conditioning on a vertex being present at a point does not affect the distribution

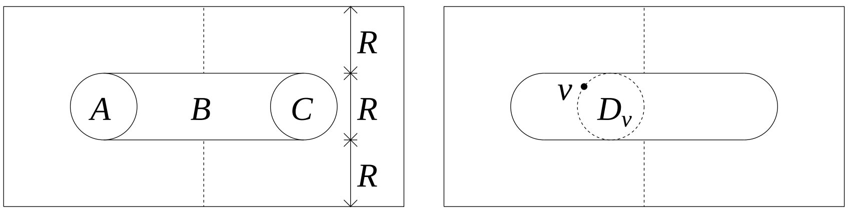

Figure 4: The regions A, B, C, and D_v in the proof of Theorem 6.

of the remaining points in a Poisson process. Thus the probability of a path of length t starting in B is at most $2|A||B|\lambda^2\, 0.842^{t-1} + \varepsilon = O(\varepsilon)$. The probability of an infinite component intersecting B is therefore $O(\varepsilon)$. Since ε is arbitrary, there is almost surely no infinite component. $\square$

Proof of Theorem 6. Assume the Poisson process has intensity λ and $R > R_0/\sqrt{\lambda}$ where $R_0 = 2.59$. We fix $p = (R_0/R)^2\lambda^{-1}$, so that G_p is given by a Poisson process with intensity $(R_0/R)^2$. By scaling $\mathbb{R}^2$ we may assume without loss of generality that G_p has intensity 1 and that $R = R_0$. Divide $\mathbb{R}^2$ into $3R \times 3R$ squares, and identify each square with an element of $\mathbb{Z}^2$ in the obvious way. We shall define a bond percolation process on $\mathbb{Z}^2$ so that percolation in this process implies percolation in $G_p^{\leq 36}$. The bonds in this percolation will not be independent, but will be 1-*independent* — any two sets S_1 and S_2 of bonds that are at graph distance at least 1 in $\mathbb{Z}^2$ from each other are independent. By Theorem 2 of [1], if the probability of each bond being open in a 1-independent process on $\mathbb{Z}^2$ is at least 0.8639, then with positive probability there is an infinite open path from the origin in $\mathbb{Z}^2$.

Consider two neighboring squares corresponding to an edge xy of $\mathbb{Z}^2$. Let A and C be disks of diameter R in the center of the squares corresponding to x and y respectively. Let B be the set of points lying directly between A and C (see Figure 4). Let $\mathcal{E}_{xy}$ be the event that every vertex of G_p in the region $A \cup B \cup C$ has at most 36 neighbors, there exists at least one vertex of G_p in A, and every vertex $v \in A \cup B$ has a neighbor in D_v, where D_v is the disk of diameter R in $A \cup B \cup C$ with v on its leftmost boundary. The expected number of $v \in A \cup B \cup C$ with more than 36 neighbors in G_p is

$$|A \cup B \cup C|\, \mathbb{P}(\mathrm{Pois}(\pi R^2) > 36) = (3 + \tfrac{\pi}{4})R^2\, \mathbb{P}(\mathrm{Pois}(\pi R^2) > 36)$$

where $\mathrm{Pois}(\mu)$ is a Poisson variable with mean μ. The probability that there is no vertex of G_p in A is $e^{-\pi R^2/4}$. The expected number of $v \in A \cup B$ with no vertex in D_v is $|A \cup B| e^{-\pi R^2/4} = 3R^2 e^{-\pi R^2/4}$. Thus (for $R = 2.59$)

$$\mathbb{P}(\mathcal{E}_{xy} \text{ fails}) \leq (3 + \tfrac{\pi}{4})R^2\mathbb{P}(\mathrm{Pois}(\pi R^2) > 36) + (1 + 3R^2)e^{-\pi R^2/4} < 0.1357.$$

Now define the bond percolation process in $\mathbb{Z}^2$ by declaring xy open if $\mathcal{E}_{xy}$ holds. If $\mathcal{E}_{xy}$ holds then every vertex in A is joined by a path in $G_p^{\leq 36}$ to one (and hence every) point in C, and there is at least one vertex in each of A and C. An infinite open path in $\mathbb{Z}^2$ therefore gives rise to an infinite path in $G_p^{\leq 36}$. The bond percolation on $\mathbb{Z}^2$ is 1-independent as the event $\mathcal{E}_{xy}$ depends only on the vertices of G_p inside the union of the squares corresponding to x and y. Since $1 - 0.1357 > 0.8639$, Theorem 2 of [1] implies that $G_p^{\leq 36}$ almost surely has an infinite component. □

6 Percolation on a lattice

In this section we shall prove Theorem 7. We shall follow the method used by Cox and Durrett [2] to prove upper bounds for oriented percolation in high dimensions. We shall need to modify this approach somewhat, since we are dealing with unoriented percolation as well as interference. The basic idea is show that there are long paths by bounding below the expected number of, and above the variance of, the number of paths of length k. Then with reasonable probability a path exists. To deal with the case when the paths are unoriented, we shall apply this method twice. The first time we shall show that with reasonable probability there are a large number of 'short' unoriented paths in $d_0 = d - 8$ dimensions. To avoid combinatorial problems associated with these paths looping back on themselves, we shall insist that they at move at most one step in any dimension (although they can go either direction in each dimension). Since we will quickly run out of dimensions, the length k of these paths will be very small compared with d. We then show that with reasonable probability we can extend some of these paths by three steps in each of the remaining 8 dimensions. Having done this, we repeat the process starting at these 8 new locations. Continuing in this manner, we show that we can couple the process with an oriented independent site percolation in $\mathbb{Z}^8$. We then use the arguments of [2] a second time to bound the critical probability of this process.

Let $\{e_1, \ldots, e_d\}$ be the standard basis of unit vectors for $\mathbb{Z}^d$. Fix integers $d_0 = d - 8$ and k and let $\mathcal{P}$ be the set of paths $P = (v_0, v_1, \ldots, v_k)$ where $v_0 = 0 \in \mathbb{Z}^d$, $v_i = v_{i-1} \pm e_{d_i}$ and the d_i are distinct. In other words, P is a path in $\{-1, 0, 1\}^{d_0} \times \{0\}^8$ from the origin such that the ℓ_1-distance from v_0 to v_i is exactly i. Fix p and let each vertex of $\mathbb{Z}^d$ be open independently with probability p.

Lemma 11. *Conditioned on the event that $P_0 = (v_0, \ldots, v_k) \in \mathcal{P}$ is open, the probability that P_0 is the unique open path of $\mathcal{P}$ from v_0 to v_k is at least $1 - 4kp$.*

Proof. We may assume without loss of generality that $v_i = e_1 + e_2 + \cdots + e_i$ for $0 \leq i \leq k$, so that $v_i = v_{i-1} + e_i$. Any other path in $\mathcal{P}$ with endpoint v_k must be of the form $v_i = v_{i-1} + e_{\pi(i)}$ for some permutation π of the numbers $\{1, \ldots, k\}$. The conditional probability that this path is open is just p^r where r is the number of vertices in the path specified by π that are not in P_0. We now count the number of permutations π which give r new vertices. The permutation π can be specified by specifying the choice of steps t at which these new vertices occur, and for each such t the value of $\pi(t)$. The values of $\pi(t)$ at all the other steps are then determined, since the vertex reached must be the same as for P_0. There are $\binom{k-1}{r}$ choices for these steps. At the ith such step t, $\pi(t)$ must be chosen so that no s with $s > t+(r-i+1)$ occurs in the set $\{\pi(1), \ldots, \pi(t)\}$, otherwise the next $r-i+1$ steps will also lead to new vertices, giving more than r new vertices in total. Thus $\{\pi(1), \ldots, \pi(t)\} \subseteq \{1, \ldots, t+(r-i+1)\}$. Hence, given $\pi(1), \ldots, \pi(t-1)$, there are at most $r-i+2$ remaining choices for $\pi(t)$ and so the total number of choices for π is at most $\binom{k-1}{r}(r+1)! \leq (r+1)k^r$. Hence the expected number of such open paths (other than P_0 itself) is at most

$$\sum_{r=1}^{\infty}(r+1)k^r p^r = \frac{kp}{(1-kp)^2} + \frac{kp}{1-kp},$$

which is at most $(\frac{16}{9} + \frac{4}{3})kp < 4kp$ when $4kp < 1$. Hence the probability that P_0 is not unique is at most $4kp$ when $4kp < 1$, and this trivially also holds when $4kp \geq 1$. $\square$

We shall set $p = 1/d$. Since each vertex of $\mathbb{Z}^d$ has $2d$ neighbors, this implies that each vertex has 2 open neighbors on average. Write $q = 1 - p$ and define

$$\alpha_{d,K} = q^{2d-K} + \binom{2d-K}{1}pq^{2d-K-1} + \binom{2d-K}{2}p^2 q^{2d-K-2}$$

to be the probability that at most 2 of some given set of $2d - K$ sites are open. Note that for any fixed K

$$\alpha_{d,K} \to \alpha_\infty := e^{-2}(1 + 2 + 2^2/2) = 5e^{-2} > 0.6766 \quad \text{as } d \to \infty.$$

Fix $P = (v_0, \ldots, v_k) \in \mathcal{P}$ and define the following events.

$\mathcal{E}_o$: P is open;

$\mathcal{E}_u$: no path $P' \in \mathcal{P} \setminus \{P\}$, joining v_0 to v_k is open;

$\mathcal{F}_i$: the vertex v_i has at most 4 open neighbors in $\mathbb{Z}^d$ $(0 \leq i < k)$;

$\mathcal{G}$: there are no open vertices $v \notin \mathbb{Z}^{d_0} \times \{0\}^8$ with $\|v - v_k\|_1 = 2$ and $\|v - v_0\|_1 = k$.

Let I_P be the indicator function of the event that P satisfies all these conditions. Note that we don't insist that v_k has few open neighbors. The set of points that we require to be closed for $\mathcal{G}$ to hold has cardinality $16k$ as there are 16 directions from v_k out of the hyperplane $\mathbb{Z}^{d_0} \times \{0\}^8$, but then we must move one step closer to $v_0 = 0$ and so must reverse one of the k steps taken along the path P from v_0 to v_k.

Let S be a set of at most d^5 points in $\mathbb{Z}^{d_0} \times \{0\}^8$, all of which are at ℓ_1-distance at least 6 from v_0. Let $X_S = \sum_{P\in\mathcal{P},\, P\cap S=\emptyset} I_P$, so that X_S is a lower bound on the number of vertices reached in k steps from 0 by open paths avoiding the set S. Let $\mathcal{H}$ be the event that both vertices 0 and $-e_d$ are open in $\mathbb{Z}^d$, and that $-e_d$ has no open neighbors in $\mathbb{Z}^d$ other than 0 and $-2e_d$. The next lemma shows that our model grows rapidly in the first few steps.

Lemma 12. *Fix $p = 1/d$ and $k = \lfloor 40 \log d \rfloor$. Then for sufficiently large d, $\mathbb{P}(X_S \geq d^{12} \mid \mathcal{H}) \geq 0.1766$.*

Proof. We shall deduce the result from bounds on the first and second moments of X_S. We first find a lower bound for $\mathbb{E}(I_P \mid \mathcal{H})$. Now $\mathbb{P}(\mathcal{E}_o \mid \mathcal{H}) = p^k$ and $\mathbb{P}(\mathcal{G} \mid \mathcal{E}_o \cap \mathcal{H}) = q^{16k}$ as $\mathcal{E}_o$ requires k additional vertices to be open and $\mathcal{G}$ requires a disjoint set of $16k$ vertices to be closed, none of which are affected by $\mathcal{H}$. Also $\mathbb{P}(\mathcal{F}_i \mid \mathcal{E}_o \cap \mathcal{H}) \geq \alpha_{d,2}$ as all vertices v_i have two open neighbors on path, so can have at most 2 more open neighbors out of $2d-2$ remaining neighbors. (For $i = 1$ there is an inequality as $\mathcal{H}$ implies that one neighbor of v_1 is known to be closed.) Lemma 11 implies that $\mathbb{P}(\mathcal{E}_u \mid \mathcal{E}_o \cap \mathcal{H}) \geq 1 - 4kp$. Note that $\mathcal{E}_u$ depends only on the status of vertices in $\mathbb{Z}^{d_0} \times \{0\}^8$, so conditioning on $\mathcal{H}$ is equivalent to conditioning on v_0 being open, which is included in the event $\mathcal{E}_o$. Now conditioning on $\mathcal{E}_o$ and $\mathcal{H}$ (which just fixes the states of a certain set of vertices), $\mathcal{E}_u$, $\mathcal{F}_i$, and $\mathcal{G}$ are all monotone events, decreasing as we increase the set of open vertices. Thus by the FKG inequality,

$$\mathbb{P}(I_P = 1 \mid \mathcal{E}_o \cap \mathcal{H}) = \mathbb{P}(\mathcal{E}_u \cap \textstyle\bigcap_i \mathcal{F}_i \cap \mathcal{G} \mid \mathcal{E}_o \cap \mathcal{H}) \geq (1-4kp)\alpha_{d,2}^k q^{16k}.$$

Thus as $k = o(d)$ and $q = 1 - 1/d$,

$$\mathbb{E}(I_P \mid \mathcal{H}) \geq (1-4kp)\alpha_{d,2}^k q^{16k} p^k = (1-o(1))(p\alpha_{d,2})^k.$$

Now

$$|\mathcal{P}| = 2^k d_0(d_0-1)\dots(d_0-k+1) = (2d)^k \prod_{i=0}^{k-1} \left(1 - \tfrac{i+8}{d}\right) = (2d)^k \Big(1 - O\big(\tfrac{k^2}{d}\big)\Big).$$

Also, the probability of a uniformly chosen random path in $\mathcal{P}$ meeting a specified vertex $v \in S$ is $\ell!/(2^\ell d_0(d_0-1)\dots(d_0-\ell+1))$ where $\ell = \|v - v_0\|_1$. As $6 \leq \ell \leq k = o(d)$, this probability is decreasing exponentially with ℓ, and so is at most $O(d^{-6})$. Thus at most a fraction $O(|S|d^{-6}) = O(d^{-1})$ of the paths in $\mathcal{P}$ meet S. Hence

$$\mathbb{E}(X_S \mid \mathcal{H}) \geq (1 - O(1/d))|\mathcal{P}|(1-o(1))(p\alpha_{d,2})^k = (1-o(1))(2\alpha_{d,2})^k. \tag{20}$$

Now we find an upper bound on the second moment of X_S. Fix $P_0 = (v_0, \dots, v_k) \in \mathcal{P}$ avoiding S and consider $\mathbb{E}(I_P \mid I_{P_0} = 1 \text{ and } \mathcal{H})$. If $I_P = I_{P_0} = 1$ and $P = (u_0, \dots, u_k)$, then there exists an r such that $u_i = v_i$ for $i \le k - r$ and $u_i \neq v_i$ for $i > k - r$. (Recall that $I_{P_0} = 1$ implies there is a *unique* open path from v_0 to v_k, so P cannot rejoin P_0 after they have separated.) Any two distinct vertices in $\mathbb{Z}^d$ have at most two common neighbors. Thus for $k - r < i < k$, u_i has at least $2d - 8$ neighbors that are not neighbors of any other vertex of $P_0 \cup \{-e_d, u_0, \dots, u_{i-1}\}$, as u_i can only have a common neighbor with u_{i-2}, $v_{i\pm 2}$ or v_i. Let α be the probability that u_i has at most 4 open neighbors conditioned on $I_{P_0} = 1$, $\mathcal{H}$, the openness of P, and the state of the neighbors of $-e_d, u_0, \dots, u_{i-1}$. Then $\alpha \le \alpha_{d,8}$ since we know the two neighbors on P are open, and so at most 2 of the $2d - 8$ unconditioned neighbors must be open. As $u_i \neq v_i$ for $i > k - r$ we have

$$\mathbb{E}(I_P \mid I_{P_0} = 1 \text{ and } \mathcal{H}) \le p^r \alpha_{d,8}^{r-1}.$$

There are at most $(2d_0)^r \le (2d)^r$ possible choices for P having this intersection with P_0, and $p = 1/d$, so

$$\mathbb{E}(X_S \mid I_{P_0} = 1 \text{ and } \mathcal{H}) \le \frac{1}{\alpha_{d,8}} \sum_{r=0}^{k} (2dp\alpha_{d,8})^r \le \frac{(2\alpha_{d,8})^k}{\alpha_{d,8}(1 - \frac{1}{2\alpha_{d,8}})} = \frac{(2\alpha_{d,8})^k}{\alpha_{d,8} - \frac{1}{2}}. \tag{21}$$

Now, conditioned on $\mathcal{H}$, $\mathbb{E}(X_S^2) = \sum_{P,P_0} \mathbb{E}(I_P I_{P_0}) = \sum_{P_0} \mathbb{E}(X_S \mid I_{P_0} = 1)\mathbb{P}(I_{P_0} = 1)$. Thus combining (21) and (20) we have

$$\mathbb{E}(X_S^2 \mid \mathcal{H}) \le \frac{(2\alpha_{d,8})^k}{\alpha_{d,8} - \frac{1}{2}} \mathbb{E}(X_S \mid \mathcal{H}) \le \frac{1 + o(1)}{\alpha_{d,8} - \frac{1}{2}} \Big(\frac{\alpha_{d,8}}{\alpha_{d,2}}\Big)^k \mathbb{E}(X_S \mid \mathcal{H})^2 = \frac{1 + o(1)}{\alpha_\infty - \frac{1}{2}} \mathbb{E}(X_S \mid \mathcal{H})^2,$$

where we have used $k = o(\sqrt{d})$ and $\alpha_{d,8}/\alpha_{d,2} = 1 + O(1/d)$. Now $k = \lfloor 40 \log d \rfloor$, so for sufficiently large d, $\log \mathbb{E}(X_S \mid \mathcal{H}) \ge (40 \log(2\alpha_\infty) + o(1)) \log d \ge 12.1 \log d$, so $\mathbb{E}(X \mid \mathcal{H})/d^{12} \to \infty$ as $d \to \infty$.

For any non-negative random variable X and $\varepsilon > 0$ we have $\mathbb{E}(X I_{\{X > \varepsilon \mathbb{E}X\}}) \ge (1 - \varepsilon)\mathbb{E}X$, and by Cauchy-Schwarz $\mathbb{E}(X^2 I_{\{X > \varepsilon \mathbb{E}X\}})\mathbb{P}(X > \varepsilon \mathbb{E}X) \ge \mathbb{E}(X I_{\{X > \varepsilon \mathbb{E}X\}})^2$. Hence $\mathbb{P}(X > \varepsilon \mathbb{E}X) \ge (1 - \varepsilon)^2 (\mathbb{E}X)^2 / \mathbb{E}(X^2)$. Thus

$$\liminf_{d \to \infty} \mathbb{P}(X_S > d^{12} \mid \mathcal{H}) \ge \alpha_\infty - \tfrac{1}{2} > 0.1766.$$

The result follows. □

Proof of Theorem 7. The fact that $\mathbb{Z}^d$ does not 3-percolate follows from Theorem 1 for all $d \ge 3$. Thus it remains to prove that for sufficiently large d, $\mathbb{Z}^d$ does 4-percolate.

We couple the process with one that dominates an independent oriented site percolation on $\mathbb{Z}^8$. An *oriented* site percolation is one where one requires an infinite path of open sites in which each step of the path is obtained by increasing one of the coordinates of the point. Each site $x = (x_1, \dots, x_8) \in (\mathbb{Z}_{\geq 0})^8$ will correspond to a d_0-dimensional subspace H_x of $\mathbb{Z}^d$ where the last 8 coordinates are fixed to be the coordinates of $3x$. Since the distance in $\mathbb{Z}^d$ between these regions is always at least 3, they are independent in the interference percolation model. The coupling will proceed one site at a time, processing all sites in the layer $\sum_{i=1}^{8} x_i = \ell$ before starting with sites in the next layer $\sum_{i=1}^{8} x_i = \ell + 1$. The order in which the sites within one layer are processed is not important. At each (reachable) site x, we shall define two events. The event $\mathcal{F}'_x$ will be called the event that x is *pre-open*. The event $\mathcal{F}_x \subseteq \mathcal{F}'_x$ will be called the event that x is *open*. If there is an oriented path of open sites in $\mathbb{Z}^8$, i.e., sites x for which $\mathcal{F}_x$ holds, then an infinite path will occur in the original interference percolation model. Also, conditioned on all previous sites, $\mathcal{F}'_x$ will occur with probability $1 - o(1)$ and $\mathcal{F}_x$ will occur with probability at least $0.1766 - o(1)$. Every pre-open site x will have an *origin* $O_x \in H_x$ and a predecessor $O_x^- = O_x - e_{d_x}$ defined, where $d_x > d_0$. If x is pre-open then O_x and O_x^- will be open and the only open neighbors of O_x^- will be O_x and $O_x - 2e_{d_x}$. There will also be a set $S_x \subseteq H_x$ of *bad* vertices, where $|S_x| \leq d^5$ and all vertices of S_x are at distance at least 6 from O_x. To start with, when $x = 0$, $S_x = \emptyset$ and O_x and $O_x^- = O_x - e_{d_x}$ will be any vertices in $H_x = H_0$ satisfying these conditions. Note that as H_x is infinite, such vertices will exist almost surely.

Now suppose $x \in \mathbb{Z}^8$ is pre-open. Then with probability at least $0.1766 - o(1)$, $X_{S_x} \geq d^{12}$. The event $\mathcal{F}_x$ will be the event that this holds (and of course that x is pre-open). Assume this occurs, so there are at least d^{12} points in H_x joined to O_x by open paths. Each such point can be within distance 6 of at most $(2d)^6$ others. Hence there is a set $Y = \{y_1, y_2, \dots, y_N\}$ of at least $d^{12}/(2d)^6 = d^6/2^6$ endpoints $y_i \in H_x$ of paths counted by X_{S_x}, each at least distance 7 from the others. Indeed, Y may be constructed greedily by taking y_i to be any such vertex at distance at least 7 from $\{y_1, \dots, y_{i-1}\}$. In fact, points of Y must be at distance at least 8 apart since they are all distance k from O_x and $\mathbb{Z}^d$ is bipartite. Consider one such y_i and fix a direction $d' = d_0 + j$, $j \in \{1, \dots, 8\}$. Then with probability $p^3 = d^{-3}$ we can extend the path from y_i three steps in the positive d'-direction via open vertices $y_i + te_{d'}$, $t = 1, 2, 3$, to arrive at a vertex of $H_{x'}$, $x' = x + e_j$. The event $X_{S_x} \geq d^{12}$ depends only on the status of vertices within distance k of O_x, and those points that are neighbors of $y_i + te_{d'}$ and are within distance k of O_x are assumed to be closed, except for those vertices along the open path. Indeed, there are no such neighbors for $t = 2$, the neighbors for $t = 1$ are closed by event $\mathcal{G}$ of the definition of I_P above, and the only neighbor for $t = 0$ is the predecessor vertex along the open path. Thus with probability at least $q^{3(2d-2)} = \Theta(1)$, $y_i + te_{d'}$ has no open neighbors except for those on the path for $t = 0, 1, 2$. Note that we make no assumption on the neighbors of $y_i + 3e_{d'}$. Consider the set of directions $d' = d_0 + j$ for which $H_{x'}$, $x' = x + e_j$, has not yet been 'reached', that is, no vertex $x' - e_k$, $k \neq j$, has previously been declared open. There are at most 8 such directions. Pick disjoint subsets Y_j of Y for each such direction with $|Y_j| = d^4/2 \gg d^3$. Then with probability $1 - o(1)$ one can find $y_{i_j} \in Y_j$ so that y_{i_j} has such a path in direction $d_0 + j$. This will be the event $\mathcal{F}'_{x'}$ that $x' = x + e_j$ is pre-open. For each $y_i \in Y_j$ that failed, place $y_i + 3e_{d'}$ and all its neighbors in the set $S_{x'}$ of bad vertices assigned to the vertex x'. At most $(1 + (2d - 16))d^4/2 \leq d^5$ such bad vertices will be generated. We define $O_{x'} = y_{i_j} + 3e_{d_0+j}$ and $O^-_{x'} = y_{i_j} + 2e_{d_0+j}$. The only elements of $H_{x'}$ for which we have any information about their neighbors all lie in $\{O_x\} \cup S_{x'}$ and every point of $S_{x'}$ is at distance at least 6 (actually at least 7) from O_x. Thus when determining the openness of x' we shall not use any information about the status of vertices already encountered, except for those guaranteed by the pre-openness of x'. As the percolation in $\mathbb{Z}^8$ is oriented, paths out of $H_{x'}$ will not have neighbors from previous steps. Two paths out of H_x do not share neighbors as they originate from different $y_i \in Y$. Also note that $H_{x'}$ is given just one chance to be pre-open. If this fails we do not try to construct paths from other vertices to $H_{x'}$.

It is clear that for any vertex $x \in \mathbb{Z}^8$ that is reachable from the origin by an open path in the oriented percolation defined above, there is some vertex in H_x that is reachable from O_0 in the interference percolation model $(\mathbb{Z}^d)_p^{\leq 4}$. Thus the existence of an infinite open directed path in $\mathbb{Z}^8$ implies the existence of an infinite path in $(\mathbb{Z}^d)_p^{\leq 4}$. As all (reachable) sites in $\mathbb{Z}^8$ are open with probability at least $0.1766 - o(1)$, even conditioned on the state of all previous sites, the open sites stochastically dominate an independent oriented site percolation model on $\mathbb{Z}^8$ with site probability $0.1766 - o(1)$. The result now follows from the following lemma. $\square$

Lemma 13. *If p_c is the critical probability for oriented site percolation in $\mathbb{Z}^8$, then $p_c < 0.1735$.*

Proof. Following the proof in [2] we have $p_c \leq \rho_8$, where ρ_d is defined as the probability that two uniformly chosen random paths from the origin in $\mathbb{Z}^d$ meet at a vertex other than the origin. (The proof in [2] is for bond percolation, but a trivial modification gives the above result for site percolation.) We estimate ρ_d by fixing a path $P_0 = (v_0, v_1, \dots)$ and calculating the probability that another path $P = (u_0, u_1, \dots)$ meets this at $u_i = v_i$, $i > 0$. Then

$$\rho_d = \sum_{k=1}^{\infty} \mathbb{P}(u_k = v_k \text{ and } u_i \neq v_i, 0 < i < k).$$

Now $\mathbb{P}(u_k = v_k \mid P_0) = d^{-k} \frac{k!}{k_1! k_2! \dots k_d!}$ where $v_k = (k_1, k_2, \dots, k_d)$ and $\sum_{i=1}^d k_i = k$. This is maximized when the k_i are as equal as possible. For $i = 1, 2$ we have exact bounds of $\mathbb{P}(u_1 = v_1) = d^{-1}$ and $\mathbb{P}(u_2 = v_2,\ u_1 \neq v_1) = d^{-2} - d^{-3}$. Thus

$$\rho_d \leq d^{-1} + d^{-2} - d^{-3} + \sum_{k=3}^{\infty} \frac{d^{-k} k!}{\lfloor \frac{k}{d} \rfloor! \lfloor \frac{k+1}{d} \rfloor! \dots \lfloor \frac{k+d-1}{d} \rfloor!}$$

This series converges for $d \geq 4$, and for $d = 8$ gives $p_c \leq \rho_8 < 0.1735$. $\square$

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