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Dependent percolation in two dimensions

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Abstract. For a natural number k, define an oriented site percolation on $\mathbb{Z}^2$ as follows. Let x_i, y_j be independent random variables with values uniformly distributed in $\{1, \ldots, k\}$. Declare a site $(i, j) \in \mathbb{Z}^2$ *closed* if $x_i = y_j$, and *open* otherwise. Peter Winkler conjectured some years ago that if $k \geq 4$ then with positive probability there is an infinite oriented path starting at the origin, all of whose sites are open. I.e., there is an infinite path $P = (i_0, j_0)(i_1, j_1)\cdots$ such that $0 = i_0 \leq i_1 \leq \cdots$, $0 = j_0 \leq j_1 \leq \cdots$, and each site (i_n, j_n) is open. Rather surprisingly, this conjecture is still open: in fact, it is not known whether the conjecture holds for *any* value of k. In this note, we shall prove the weaker result that the corresponding assertion holds in the *unoriented* case: if $k \geq 4$ then the probability that there is an infinite path that starts at the origin and consists only of open sites is positive. Furthermore, we shall show that our method can be applied to a wide variety of distributions of (x_i) and (y_j). Independently, Peter Winkler [14] has recently proved a variety of similar assertions by different methods.

1. Introduction

In the most widely-studied percolation models (see for example [6]), the various bonds or sites behave completely independently. In many models of interest, however, this independence is lost. Some of these dependent models, such as the random-cluster model (see [7], [8]) are of substantial intrinsic interest; others, such as those arising in renormalization arguments, are used to give information about the classical independent setting (for example that studied in [1]).

In general, it can be extremely difficult to prove results about dependent models, even if the corresponding results in the independent case are almost a triviality; it is an important challenge to develop techniques for tackling these difficulties. Methods of Klein [10] apply to some dependent processes, including the contact process in a random environment – which can naturally be seen as a dependent

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percolation model. Perhaps the most basic question, for many processes of interest, is to show that there is a non-trivial phase transition; in percolation processes this normally amounts to showing that percolation occurs for some parameter values, but not for others. Where the dependencies are merely local (that is, sites or bonds are independent, or nearly independent, provided they are far apart) then general methods of [11] answer this basic question, but long-range dependencies present a far greater challenge. One of the most infamous models with long-range dependencies, for which even the most basic question – of the existence of percolation at any parameter value – remains unanswered, is due to Peter Winkler.

To describe Winkler's problem precisely, and the related problem that we shall address, we consider the 2-dimensional lattice $\mathbb{Z}^2$. Let $x_i, y_j \in \{1 \dots k\}$, be two sequences of symbols or *colors*. The sequences may be finite (e.g., defined for $1 \le i, j \le n$), infinite (e.g., defined for $i, j \in \mathbb{N}$) or doubly infinite (defined for all $i, j \in \mathbb{Z}$). Declare a site $(i, j) \in \mathbb{Z}^2$ to be *closed* or *blocked* if $x_i = y_j$, otherwise declare it to be *open*. Sites where x_i or y_j are not defined are also considered open. A *trail* is a sequence $(i_0, j_0), (i_1, j_1) \cdots$ of sites in $\mathbb{Z}^2$ each one of which is at distance exactly 1 from the previous one; it is a *path* if additionally all the sites are distinct. A path as above is said to be *oriented* if, for all n, $i_{n+1} \ge i_n$ and $j_{n+1} \ge j_n$. A path or trail is *traversable* or *open* if each of the sites (i_n, j_n) is open.

About ten years ago, Peter Winkler conjectured that if x_i, y_j are independent random variables uniformly distributed in $\{1, \dots, k\}$, where $k \ge 4$, then there is a positive probability of an infinite open oriented path starting at the origin. (In fact, this formulation of the conjecture is due to Noga Alon: Winkler formulated his conjecture in terms of colliding random walks on graphs involving 'clairvoyant demons'. For some results on colliding random walks, see [3].) This fascinating and notorious conjecture is still open; moreover, there is no value of k for which the probability of survival is known to be strictly positive.

The aim of this paper is to decide under what circumstances we have strictly positive probability of survival (percolation) in the *unoriented* case. Of course, we say that a site (i, j) *survives* if there is a traversable trail starting from (i, j) with an infinite number of vertices. By a simple compactness argument, this is equivalent to the existence of traversable trails reaching sites arbitrarily far from the origin.

Now assume x and y are two independent random sequences drawn from two distributions μ_x and μ_y (which may be the same distribution). We shall consider the probability $p_s(i, j)$ that the site (i, j) survives. We shall mainly be interested in the case when x and y are doubly infinite random sequences and both distributions are translation invariant. Under these circumstances $p_s = p_s(i, j)$ will be independent of the site (i, j). The question arises, is p_s strictly positive? One special case is when x_i, y_j are independent random variables uniformly distributed in $\{1, \dots, k\}$, for which we shall show that p_s is strictly positive precisely when $k \ge 4$. We shall also prove results for more general distributions including the case when x_i and y_j come from different distributions. In particular, we shall give conditions on the distribution of one of these sequences that ensure that the probability of survival is strictly positive, provided the other sequence is independent and its distribution is translation invariant. Independently, similar results have been proved by Winkler [14] in the case when the two sequences have identical distributions.

Despite these results, the original problem of oriented percolation seems far from being proved. Indeed, a recent result of Gács [5] shows that if the probability of percolation to infinity is positive then the probability that there is percolation to distance n but not to infinity does *not* decrease exponentially with n. This is in contrast to the unoriented case and to most of the other percolation problems that have been studied to date. Since many of the standard percolation arguments rely on the probability of surviving to distance n but not to infinity decreasing exponentially with n, this suggests that the oriented case is very hard.

2. Preliminary results

We shall call a sequence u *non-repeating* if for all i, $u_i \neq u_{i-1}$. We shall call a sequence u *compressed* if for all i, both $u_i \neq u_{i-1}$ *and* $u_i \neq u_{i-2}$. Often, we shall not be interested in the starting index of a finite sequence, and so we consider finite sequences to be *words* $u_1 \dots u_n$, i.e., ordered n-tuples of colors.

In the percolation process, we can assume that the sequences x and y are non-repeating, since for example if $x_i = x_{i-1}$, we can delete the column i without affecting the survivability of the other vertices. To delete column i, simply delete all edges between $(i-1, j)$ and (i, j) in any traversable trail and identify columns i and $i-1$. Conversely, to reinstate column i, add in an edge between $(i-1, j)$ and (i, j) whenever the trail crosses between columns $i-1$ and $i+1$. Similarly if $y_j = y_{j-1}$ we can delete row j. Unfortunately, this process of deleting rows and columns may destroy the translation invariance of the distributions of x and y, so in most of what follows we shall *not* assume the sequences are non-repeating.

Lemma 1. *If the sequence x is compressed, then every open site survives.*

Proof. Suppose we have constructed a traversable path to (i, j). We shall construct a path up to the next row. If $(i, j+1)$ is open then extend the path to this site. Otherwise, $x_{i-1} \neq x_{i+1}$ and either $(i-1, j)$ or $(i+1, j)$ is open. Without loss of generality assume that $(i+1, j)$ is open. Now since $x_i \neq x_{i+1}$ and $(i, j+1)$ is blocked, $(i+1, j+1)$ must be open. We can therefore move right one step and up one step to get there. Repeating this process we can reach at least one site in every row above our initial row. □

Note that we made no assumption about the sequence y in this lemma, however by symmetry, the result also holds if y is compressed and x is arbitrary.

For a finite sequence (word) $u = u_1 \dots u_n$, let its *compression* $A(u)$ be the word defined inductively by $A(u_1) = u_1$ and if $A(u_1 \dots u_r) = a_1 \dots a_s$ then

$$A(u_1 \dots u_{r+1}) = \begin{cases} a_1 \dots a_s & \text{if } u_{r+1} = a_s, \\ a_1 \dots a_{s-1}, & \text{if } u_{r+1} = a_{s-1} \text{ and } s > 1, \\ a_1 \dots a_s u_{r+1}, & \text{otherwise.} \end{cases} \tag{1}$$

Let a and b be two colors. An *aa-compression* of a word u is a word obtained by replacing some occurrence of aa in u with a. An *aba-compression* of u is a word obtained by replacing some occurrence of aba in u with a. A *simple compression* will be any compression of either of these types. If u is a sequence, define u^* to be the reverse of u, i.e., $(u^*)_i = u_{-i}$. If u and v are finite sequences, let uv be the sequence obtained by appending v to u. For finite sequences u, write $|u|$ for the length of u. Note in particular that $|A(u)| \le |u|$.

Starting with a word u and repeatedly applying simple compressions, we eventually arrive at a compressed word, i.e., one that has no further simple compressions. As the lemma below shows, no matter in which order we apply simple compressions, the resulting compressed word is precisely $A(u)$.

Lemma 2. *For every (finite) word u,*

1. *the word $A(u)$ is compressed,*
2. *u is compressed if and only if $A(u) = u$,*
3. *if u' is any sequence obtained from u by a simple compression then $A(u') = A(u)$,*
4. *$A(u)$ can be obtained from u by a finite number of simple compressions,*
5. *$A(u^*) = A(u)^*$.*

Proof.

1. Is clear from the definition of $A(u)$.
2. By 1, it is enough to prove that if u is compressed then $A(u) = u$. This follows by induction, since if $A(u_1 \dots u_s) = u_1 \dots u_s$ and $u_{s+1} \ne u_s, u_{s-1}$ then $A(u_1 \dots u_{s+1}) = u_1 \dots u_{s+1}$.
3. Say $u = vaaw$ or $vabaw$ and $u' = vaw$ for some words v and w. It is enough to show $A(vaa) = A(vaba) = A(va)$, since then by construction, $A(u) = A(u')$. Now $A(u)$ always starts and finishes with the same colors as u, so we can assume $A(va) = sa$ for some word s and $A(vaa) = sa = A(va)$. If s ends in b then $A(vab) = s$ and $A(vaba) = sa = A(va)$ (s does not end in ab by part 1). Otherwise $A(vab) = sab$ and $A(vaba) = sa = A(va)$.
4. If u is not compressed then for some i, $u_i = u_{i-1}$ or $u_i = u_{i-2}$. Therefore, there exists an aa-compression or aba-compression for some a and b. By 3, we can replace u by such a compression. Using part 2 and induction on the length of u gives the result.
5. Simple compressions are symmetric in the sense that if u' is a simple compression of u then $(u')^*$ is a simple compression of u^*. Hence, using part 4, $A(u)^*$ can be obtained from u^* by a sequence of simple compressions. Part 3 now implies $A(u^*) = A(A(u)^*)$. However, since $A(u)^*$ is compressed, $A(A(u)^*) = A(u)^*$.

□

Our next aim is to prove the triangle inequality for compressions of concatenations.

Lemma 3. *If u and v are two finite sequences then*

$$||A(u)| - |A(v)|| \le |A(uv)| \le |A(u)| + |A(v)|.$$

Proof. We can reduce u and v to $A(u)$ and $A(v)$ by sequences of simple compressions, so we can reduce uv to $A(u)A(v)$ by the same. By part 3 of Lemma 2, $A(A(u)A(v)) = A(uv)$, and so

$$|A(uv)| = |A(A(u)A(v))| \le |A(u)A(v)| = |A(u)| + |A(v)|. \qquad (2)$$

From the construction of $A(u)$, we see that adding a single color to u can decrease $|A(u)|$ by at most one. Hence

$$|A(uv)| = |A(A(u)A(v))| \geq |A(A(u))| - |A(v)| = |A(u)| - |A(v)|. \quad (3)$$

Finally, by reversing the sequences u and v we have

$$|A(uv)| = |A(uv)^*| = |A(v^*u^*)| \geq |A(v^*)| - |A(u^*)| = |A(v)| - |A(u)|. \quad (4)$$

□

We shall now generalise Lemma 1 to non-compressed sequences x. This is the crucial step in our argument since it gives us a simple criterion which guarantees the survival of many sites in our percolation. First we need a couple of technical lemmas.

Lemma 4. *Assume $i_1, \ldots, i_r$ is a sequence of integers with $|i_t - i_{t+1}| \leq 1$ for $1 \leq t < r$ and $n = i_1 \leq i_r = m$. If x is a sequence of colors then $A(x_{i_1}x_{i_2}\ldots x_{i_r}) = A(x_n x_{n+1}\ldots x_m)$ provided both are defined.*

Proof. Viewing $i_1 \ldots i_r$ as a sequence of colors (in $\mathbb{Z}$), it is enough to show that there exists a sequence of simple compressions of $i_1 \ldots i_r$ which gives $n, n+1, \ldots, m$. This is because we can apply $x_a x_a$ and $x_a x_b x_a$-compressions to $x_{i_1} \ldots x_{i_r}$ which correspond to the aa and aba-compressions on $i_1, \ldots, i_r$. The result will then follow by part 3 of Lemma 2. Consider $A(i_1 \ldots i_r) = a_1 \ldots a_s$. From the construction of $a_1 \ldots a_s$, we see that every consecutive pair $a_t a_{t+1}$ is a pair $i_{t'} i_{t'+1}$ of the original sequence. Since $a_t \neq a_{t+1}$, this implies $|a_t - a_{t+1}| = 1$. Now, if $a_t - a_{t+1} = 1$, say, and $a_{t+1} - a_{t+2} = -1$ then $a_{t+2} = a_t$ contradicting part 1 of Lemma 2. Hence $a_t - a_{t+1} = c$ is a constant, $c = \pm 1$. Now $a_1 = n \leq m = a_s$, so $c = 1$ and $a_1 \ldots a_s = n, n+1, \ldots, m$. Since $a_1 \ldots a_s$ is obtained from $i_1 \ldots i_r$ by a sequence of simple compressions, we are done. □

Lemma 5. *Let y be any sequence and assume j is any fixed value for which y_j is defined. If none of the sites (i, j) with $n \leq i \leq m$ survive, and if the sites $(n-1, j)$ and $(m+1, j)$ do survive, then $A(x_n \ldots x_m) = y_j$.*

Proof. Consider the set S of all sites that do *not* survive and let C be the connected component of S that contains the sites (i, j) with $n \leq i \leq m$. By connected, we mean connected by paths which travel distance 1 at each step (connected by edges rather than diagonals). Let B be the boundary of C. I.e, B consists of sites of C that are distance 1 from points not in C. The sites of B are clearly blocked and B must be at least diagonally connected (connected by paths that travel at most $\sqrt{2}$ at each step). Also $(n, j), (m, j) \in B$, so there must be a trail $(i_1, j_1), \ldots, (i_r, j_r)$ with $(i_s, j_s) \in B$, $(i_1, j_1) = (n, j)$, $(i_r, j_r) = (m, j)$ and $|i_s - i_{s+1}|, |j_s - j_{s+1}| \leq 1$. Using Lemma 4 and noting that $x_{i_s} = y_{j_s}$ (since sites in B are blocked), we get

$$A(x_n x_{n+1} \ldots x_m) = A(x_{i_1} x_{i_2} \ldots x_{i_r}) = A(y_{j_1} y_{j_2} \ldots y_{j_r}) = A(y_j) = y_j. \quad (5)$$

□

Now we give the key result we need. Recall from the introduction that if x_i is not defined then any site (i, j) is automatically open, so escaping from a certain vertical strip is sufficient for survival in Theorem 6.

Theorem 6. *Let y be any sequence and let x be any finite sequence $x = x_n \dots x_m$. Then in each row j where y_j is defined, there are at least r_j sites (i, j) with $n \le i \le m$ which survive, where r_j is the number of terms in $A(x)$ not equal to y_j.*

Proof. By Lemma 5, there are simple compressions that reduce x to a sequence x' in which each block of sites that does not survive in row j has been replaced with a single color y_j. The number of sites that do survive is just the number of the remaining terms in x'. In particular, it is at least as large as the number of terms in $A(x') = A(x)$ that do not equal y_j, since any further simple compressions of x' can only decrease this number. □

Corollary 7. *If $x = x_n \dots x_m$ is finite, then in each row j there are at least $\lfloor \frac{2}{3}|A(x)|\rfloor$ sites (i, j) with $n \le i \le m$ which survive.*

Proof. The sequence $A(x)$ is compressed, so each three successive terms must be distinct and so at most one in every three can be equal to y_j. □

3. The height function

So far everything we have done has been deterministic. It is now time to introduce randomness into the picture. Let u be a doubly infinite random sequence drawn from a distribution μ_u. We shall make the following assumption:

Hypothesis A: The distribution μ_u of u is translation invariant and the set

$$S_1 = \{(n, m) : n < 0 < m \text{ and } |A(u_n \dots u_m)| = 1\}$$

is finite with probability 1.

Our aim shall be to extend Corollary 7 to infinite sequences x, but in order to do this we shall need an analogue of the length of a compressed sequence for infinite sequences x.

Lemma 8. *If $i \le j \le l$ and $|A(u_i \dots u_j)| \le |A(u_i \dots u_k)|$ for all k with $j \le k \le l$ then*

$$|A(u_j \dots u_l)| = |A(u_i \dots u_l)| - |A(u_i \dots u_j)| + 1.$$

Proof. Let $A(u_i \dots u_j) = su_j = sA(u_j)$ for some word s. By considering the construction of $A(u)$, we see that as long as $|A(u_i \dots u_k)| \ge |A(u_i \dots u_j)|$, the word $A(u_i \dots u_k)$ will start with the word su_j and the construction of $A(u_i \dots u_k)$ will follow that of $A(u_j \dots u_k)$ with $A(u_i \dots u_k) = sA(u_j \dots u_k)$ for each k with $j \le k \le l$. Putting $k = l$ and taking the length of both sides gives the result. □

Lemma 9. *Under Hypothesis A and for any fixed n, as m tends to infinity both $|A(u_n u_{n+1} \dots u_m)|$ and $|A(u_n u_{n-1} \dots u_{-m})|$ tend to infinity almost surely.*

Proof. Assume otherwise and let $L = \liminf_{m\to\infty} |A(u_n \dots u_m)|$ so that $L < \infty$ with positive probability. If $L < \infty$, define m_0 to be any value of m such that $|A(u_n \dots u_{m_0})| = L$ and $|A(u_n \dots u_m)| \geq L$ for all $m > m_0$ (with equality for infinitely many m). If $L = \infty$ define m_0 arbitrarily. By Lemma 8, $|A(u_{m_0} \dots u_m)| = 1$ for infinitely many values of $m > m_0$ when $L < \infty$. Hence there is at least one value $c \in \mathbb{Z}$ such that, with positive probability, $m_0 = c - 1$ and the set $\{(n, m) : n < c < m \text{ and } |A(u_n \dots u_m)| = 1\}$ is infinite. Using translation invariance, the set $\{(n, m) : n < 0 < m \text{ and } |A(u_n \dots u_m)| = 1\}$ is then infinite with positive probability contradicting Hypothesis A. The case of $|A(u_n u_{n-1} \dots u_{-m})|$ follows from the first part by considering the reversed sequence. □

Now define the height function $h = h_u$ of a sequence u by

$$h_u(i) = \lim_{n\to\infty} (|A(u_{-n} \dots u_i)| - |A(u_{-n} \dots u_0)|), \tag{6}$$

whenever this limit exists. When h_u is defined, we have

$$|A(u_{j+1} \dots u_i)| \geq h_u(i) - h_u(j) = \lim_{n\to\infty} (|A(u_{-n} \dots u_i)| - |A(u_{-n} \dots u_j)|). \tag{7}$$

Lemma 10. *Under Hypothesis A, $h_u(i)$ exists almost surely.*

Proof. Assume first that $i \geq 0$. Since $|A(u_0 \dots u_{-n})| \to \infty$, there exists an N such that for $n \geq N$ this length is at least $i + 1$. From the construction of the sequence $A(u)$, we see that the first $i+1$ terms of $A(u_0 \dots u_{-n}) = a_0 a_1 \dots$ are independent of n for $n \geq N$. Therefore, for any $n \geq N$, $A(u_{-n} \dots u_0)$ ends in $a_i \dots a_0$ by part 5 of Lemma 2. Looking at the construction of $A(u_{-n} \dots u_i)$, the choices of when to add or remove terms going from $A(u_{-n} \dots u_0)$ to $A(u_{-n} \dots u_i)$ can depend only on the last $i + 1$ terms of $A(u_{-n} \dots u_0)$. Therefore $|A(u_{-n} \dots u_i)| - |A(u_{-n} \dots u_0)|$ is independent of n for $n \geq N$, implying the result.

For $i < 0$ the argument is similar: we just interchange i and 0 and use the fact that $|A(u_i \dots u_{-n})| \to \infty$ as $n \to \infty$. □

Define the random variable $\alpha_u = \lim_{|n|\to\infty} h_u(n)/n$ when this limit exists.

Lemma 11. *Under Hypothesis A, the random variable α_u exists, is strictly positive and is equal to the limit $\lim_{n\to\infty} \frac{1}{n}|A(u_1 \dots u_n)|$ almost surely.*

Proof.
1. Existence.
Define $f(u) = h_u(1) = h_u(1) - h_u(0)$ and define the transformation T by $(Tu)_i = u_{i+1}$. By translation invariance of the distribution, T is a measure preserving transformation and f is an L^1 function. Also $f(T^r u) = h_u(r+1) - h_u(r)$, so $h_u(n) = f(u) + f(Tu) + \cdots + f(T^{n-1}u)$. By Birkhoff's Theorem (see [4] or [13]) $h_u(n)/n = (f(u) + f(Tu) + \cdots + f(T^{n-1}u))/n \to \mathbb{E}(f(u) \mid \mathscr{I})$ almost surely, where $\mathscr{I}$ is the σ-field of T-invariant events. Noting that the same argument applies to $h_u(-n)/n$ with T replaced by T^{-1}, and that the T-invariant events are precisely the same as the T^{-1}-invariant events, the limit exists as required.

2. Strictly positive.
Assume that $\alpha_u \le 0$ with positive probability. By conditioning on the event $\alpha_u \le 0$, we can assume it holds with probability 1. Since the definition of α_u is translation invariant, Hypothesis A will still hold. Now use Lemma 9 and define n_L for $L > 0$ so that $|A(u_0 \dots u_{-n_L})| = L + 1$ and $|A(u_0 \dots u_{-n})| \ge L + 1$ for all $n > n_L$. Using Lemma 8 and part 5 of Lemma 2,

$$|A(u_{-n} \dots u_{-n_L})| - |A(u_{-n} \dots u_0)| = 1 - |A(u_{-n_L} \dots u_0)| = -L \qquad (8)$$

for all $n > n_L$ and hence $h_u(-n_L) = -L$. If there exists an $m_L \ge 0$ with $h_u(m_L) = -L$, then there exist $n, m \ge 0$ with $h_u(-n) = h_u(m) = -L$, such that $h_u(i) > -L$ for all $-n < i < m$. Using Lemma 8, and the definition of h_u, we then get $|A(u_{-n} \dots u_m)| = 1$. Since this can happen for only finitely many pairs (m, n), the set $\{h_u(m) : m \ge 0\}$ must be bounded below.

Let A_i be the event that $h_u(j) > h_u(i)$ for all $j > i$. Since $\{h_u(m) : m \ge 0\}$ is bounded below but $\{h_u(-n) : n \ge 0\}$ is not, there exists some i for which A_i occurs. By translation invariance, A_i's occur for all i with the same probability p, which must be positive since some A_i do occur. If A_{i_0} occurs, then $h_u(i_1) - h_u(i_0)$ is at least as big as the number of A_is that occur with $i_0 \le i < i_1$, so one has for any $0 < K < r$,

$$h_u(r) \ge \max\Big(-r, \quad \sum_{i=1}^{r} 1_{A_i} - \inf\{|i| : -K < i < K, A_i \text{ occurs }\}\Big);$$

from which we can deduce that $\mathbb{E}(h_u(r)/r) \ge p - K/r - 2\mathbb{P}(\bigcap_{-K<i<K} A_i^c)$. Letting $r \to \infty$ then $K \to \infty$ we see that $\liminf_{r\to\infty} \mathbb{E}(h_u(r)/r) \ge p > 0$. However, $h_u(r)/r$ is bounded (by ± 1) and tends to $\alpha_u \le 0$ almost surely. Hence $\liminf \mathbb{E}(h_u(r)/r) \le 0$. This is a contradiction, so the assumption on α_u was false and $\alpha_u > 0$ almost surely.
3. Since $h_u(r) \to \infty$ almost surely, there exists some i such that $h_u(j) > h_u(i)$ for all $j > i$. The definition of h_u together with Lemma 8 then implies that $|A(u_i \dots u_j)| = h_u(j) - h_u(i) + 1$ for all $j > i$. The triangle inequality then gives $||A(u_1 \dots u_j)| - h_u(j)| \le |i| + |h_u(i)| + 1$ and the result follows. □

Corollary 12. *If x satisfies Hypothesis A, then the percolation survives from the origin with probability at least $\frac{2}{3}\mathbb{E}(\alpha_x) > 0$.*

Proof. For any $\epsilon > 0$, we can find an N such that for all $n \ge N$

$$|A(x_{-n+1} \dots x_n)| \ge h_x(n) - h_x(-n) > 2n(\alpha_x - \epsilon). \qquad (9)$$

The number of sites between $(-n+1, 0)$ and $(n, 0)$ that survive a distance at least ϵn is at least the number of sites $(i, 0)$ with $|i| \le (1-\epsilon)n$ which escape from the box $(-n, n] \times (-n, n]$. By Corollary 7 this number is at least $\frac{2}{3}(2n(\alpha_x - \epsilon)) - 2\epsilon n$. Excepting a set of measure ϵ, we can take N above to be non-random. By translation invariance, this implies the probability of any site surviving a distance at least ϵN is at least $\mathbb{E}\left(\frac{2}{3}(\alpha_x - \epsilon) - \epsilon\right) - \epsilon$. Letting N tend to infinity and ϵ tend to zero gives the result. □

Theorem 13. *If x is a symmetric random walk on $k \geq 4$ colors, then the percolation survives from the origin with probability at least $\frac{k-3}{k}$.*

Proof. We can assume that x has no repetitions. The construction of $A(x_n \dots x_m)$ shows that each time we increase m by one, the length of $A(x_n \dots x_m)$ grows by 1 with probability at least $\frac{k-2}{k-1}$ and shrinks by 1 with probability at most $\frac{1}{k-1}$. If $k \geq 4$, this implies that $|A(x_n \dots x_m)|$ can be bounded below by a random walk with positive drift $\frac{k-3}{k-1}$ as m increases. The probability that $|A(x_n \dots x_m)| = 1$ is therefore bounded above by a decreasing exponential in $m - n$ (see Appendix A) and the set $\{(n, m) : n < 0 < m \text{ and } |A(u_n \dots u_m)| = 1\}$ is therefore finite with probability 1 (see proof of Lemma 16). Hypothesis A now holds, and $\alpha_x \geq \frac{k-3}{k-1}$ by Lemma 11 (and in fact equality holds). The result now follows as in Corollary 12. We have replaced the $\frac{2}{3}$ by $\frac{k-1}{k}$ since in Corollary 7, each color is equally probable at each point in $A(x_n \dots x_m)$, so the probability of a term being equal to y_0 is only $\frac{1}{k}$. □

Once again, note that these results are independent of the values of y_j for $j \neq 0$ and require only that y_0 be independent of x.

It is clear that the probability of survival in Theorem 13 is at most $\frac{k-1}{k}$, since this is just the probability of the origin being open. Thus the probability of survival lies between $\frac{k-3}{k}$ and $\frac{k-1}{k}$. Numerical simulations for $k = 4$ and both x and y symmetric random walks on k colors suggest that the true value for the survival probability in this case is much closer to $\frac{k-1}{k}$ than $\frac{k-3}{k}$.

4. Markov chains

Assume that u is a doubly infinite sequence drawn from a translation invariant distribution μ_u. Since μ_u is translation invariant, we can define probabilities $p_a = \mathbb{P}(u_i = a)$ and $p_{ab} = \mathbb{P}(u_{i+1} = b \,|\, u_i = a)$ independently of the index i. Define the graph $G = G_u$ of μ_u by letting $V(G) = \{a : p_a > 0\}$ be the set of colors that can occur and $E(G) = \{ab : \text{either } p_{ab} > 0 \text{ or } p_{ba} > 0\}$.

Lemma 14. *If μ_x and μ_y are independent translation invariant distributions and $G_x \neq G_y$ then the origin survives with positive probability.*

Proof. Without loss of generality, we can assume $p = p_a p_{ab} > 0$ in μ_y but $p_a p_{ab} = p_b p_{ba} = 0$ in μ_x for some colors a and b. Assume first that $x_0 \neq a$. With probability p, $y_0 = a$ and $y_1 = b$, and there is an infinite open path starting at the origin going to the right along rows 0 and 1. (E.g., if we reach $(i, 0)$ say, then either $(i + 1, 0)$ is open or $x_{i+1} = a$, $x_i \neq a, b$ and so both $(i, 1)$ and $(i + 1, 1)$ are open and we can always get to the next column). Similarly if $x_0 = a$, then $y_{-1} = a$ and $y_0 = b$ with probability p. In all cases, the origin survives with probability at least p. □

From now on, assume the distribution of the sequence x is given by an ergodic Markov chain. The probability p_{ab} is now just the transition probability from state a to state b. By an ergodic Markov chain, we mean that for each pair of states a and b, there is a positive probability of reaching b from a after some number of steps. I.e., there is a sequence of states $a_1, \ldots, a_r$ with $a = a_1$, $b = a_r$ and $p_{a_s a_{s+1}} > 0$ for $1 \le s < r$. This implies that there is a unique stationary distribution and allows us to define the chain x_i for all $i < 0$.

A *traversable cycle* of $G = G_x$ is a cycle $a_0, \ldots, a_r = a_0$ of G in which either $p_{a_s a_{s+1}} > 0$ for all $s = 0, \ldots, r-1$ or for which $p_{a_{s+1} a_s} > 0$ for all $s = 0, \ldots, r-1$. I.e., a traversable cycle is a cycle of states, at least one of whose two orientations could occur in the sequence x. A cycle in G is called *unbiased* if $\prod_s p_{a_s a_{s+1}} = \prod_s p_{a_{s+1} a_s}$, i.e., the cycle occurs in x with the same probability as its reversal. A cycle is *biased* if it is not unbiased. Clearly, any biased cycle is traversable.

Let $\pi_1(G, a)$ be the fundamental group of G based at $a \in V(G)$. The fundamental group $\pi_1(G, a)$ can be represented as the group generated by possible finite subsequences u of x starting and ending with a modulo homotopy. It is clear that simple compressions are homotopies, so $A(u)$ and u represent the same element of $\pi_1(G, a)$. In fact, the converse is true: u and v represent the same element of $\pi_1(G, a)$ precisely when $A(u) = A(v)$. Composition in the group is given by concatenation of sequences.

We shall consider random walks on $H = \pi_1(G, a)$. Let d_t be i.i.d. random variables with values in H, and define a random walk w_t, $t \in \mathbb{Z}$ by setting w_0 equal to the identity element 1_H of H and defining for all t, $w_{t+1} = w_t d_t \in H$. This gives us two random walks $\{w_t : t \ge 0\}$ and $\{w_{-t} : t \ge 0\}$ starting at the identity of H. We call w_t a *bidirectional* random walk. If $H \cong \mathbb{Z}$, we say the walk has a *drift* if $\mathbb{P}(d_t = d)$ is bounded by a decreasing exponential in $|d|$ and $\mathbb{E}(d_t) \ne 0$ where d_t is considered as an element of $\mathbb{Z}$.

The sequence x gives rise to a bidirectional random walk as follows. Let $x_0 = a$ and define i_t, $t \in \mathbb{Z}$, to be the indices of the occurrences of a in x. I.e., $i_0 = 0$, $i_t < i_{t+1}$, $x_{i_t} = a$ and $x_i = a$ implies $i = i_t$ for some t. Since finite ergodic Markov chains are recurrent, with probability 1 there are an infinite number of i with $x_i = a$ and so i_t is defined for all $t \in \mathbb{Z}$. Define d_t to be the element of $\pi_1(G, a)$ represented by the sequence $x_{i_t} \ldots x_{i_{t+1}}$. Since x is a Markov chain, the d_t are i.i.d. random variables.

Lemma 15. *If G contains precisely one cycle then the bidirectional random walk on $\pi_1(G, a)$ defined above has a drift if and only if the cycle is biased.*

Proof. If there is only one cycle then $\pi_1(G, a) \cong \mathbb{Z}$. Assume the generator of $\pi_1(G, a)$ corresponding to $1 \in \mathbb{Z}$ is represented by a loop $u = u_0 \ldots u_n$ based at $u_0 = u_n = a$. This loop is contractible to the unique cycle. Let $p_+ = \prod_s p_{u_s u_{s+1}}$ and $p_- = \prod_s p_{u_{s+1} u_s}$. We can assume at least one of p_+ and p_- is non-zero. (If the cycle is not traversable then the random walk is constant). The ratio of p_+ to p_- is unchanged if we contract u, since this just corresponds to removing identical factors of the form $p_{ab} p_{ba}$ or p_{aa} from both products. (Recall that contracting corresponds to simple compressions). Therefore $p_+ = p_-$ if and only if the cycle is unbiased. Without loss of generality, assume $p_- \ge p_+$. The probability of looping round the cycle n times without hitting a decreases exponentially with n (by standard properties of finite Markov chains) so we need only check the expectation $\mathbb{E}(d_t)$. Assume there is a positive probability $p_v = \prod_s p_{v_s v_{s+1}}$ of obtaining a loop

$v = v_0 \dots v_r$ representing $n \in \mathbb{Z}$. Let p_{v^*} be the probability of obtaining the reverse of v. Contracting v and v^* by simple compressions does not change the ratio of p_v to p_{v^*}, so p_{v^*}/p_v is the same as that for n copies of u. I.e., $p_{v^*}/p_v = (p_-/p_+)^n$. If $p_- = p_+$ then $p_{v^*} = p_v$ and the probability that $d_t = n$ is the same as that for $d_t = -n$. If $p_- > p_+$ then the probability that $d_t = n > 0$ is less than or equal to the probability that $d_t = -n$ (and strictly less in the case when $n = 1$ by considering the case $v = u$). Hence $\mathbb{E}(d_t) = 0$ if and only if the cycle is unbiased. □

A bidirectional walk w_t will be called *doubly transient* if the set $S_1 = \{(n, m) : n < 0 < m \text{ and } w_n = w_m\}$ is finite with probability 1. This is a stronger condition than requiring both $\{w_t : t \geq 0\}$ and $\{w_{-t} : t \geq 0\}$ to be transient. If $x, y \in H$, we say x and y are *independent* if $x^n \neq y^m$ for all $(n, m) \in \mathbb{Z}^2 \setminus (0, 0)$. The fundamental group $\pi_1(G, a)$ is a free group (see [12], Theorem 3.3.13) and we shall use the following fact about such random walks on free groups.

Lemma 16. *Let H be a free group and w_t, $t \in \mathbb{Z}$ a bidirectional random walk as above. If d_t can take on at least two independent values, or if $H \cong \mathbb{Z}$ and w_t has a drift then w is doubly transient.*

Proof. In both cases, the probability $\mathbb{P}(w_{t+n} = w_t) = \mathbb{P}(w_n = 1_H)$ is bounded above by λ^n for some constant $\lambda < 1$ (see Appendix A). Therefore

$$\mathbb{E}(|S_1|) = \sum_{n<0<m} \mathbb{P}(w_n = w_m) \leq \sum_{n<0<m} \lambda^{m-n} = \lambda^2/(1-\lambda)^2 < \infty. \tag{10}$$

Hence, S_1 is finite almost surely. □

Corollary 17. *If μ_x is an ergodic Markov chain and G_x contains at least two traversable cycles or G_x contains only one cycle and this cycle is biased, then x satisfies Hypothesis A.*

Proof. First we shall show the bidirectional random walk defined above satisfies the conditions of Lemma 16. If there are at least two traversable cycles, then $\pi_1(G, a)$ has free rank at least two, and there are at least two independent elements of this group that can occur as increments d_t in the random walk. On the other hand, if G has precisely one cycle which is biased, then the random walk is on $\mathbb{Z}$ and has a drift by Lemma 15. We now know that the walk is doubly transitive by Lemma 16. Assume Hypothesis A fails. Since μ_x is translation invariant, this implies that the set $\{(n, m) : n < 0 < m \text{ and } |A(u_n \dots u_m)| = 1\}$ is infinite with positive probability. In particular, there must be some color a such that $|A(u_n \dots u_m)| = 1$ and $A(u_n \dots u_m) = a$ occurs for infinitely many (n, m) with positive probability. Conditioning on this, there must be some $c \in \mathbb{Z}$ with $u_c = a$ and $n < c < m$ for infinitely many (n, m) with positive probability. By translation invariance we may assume $c = 0$ and $a = u_0$. For these pairs (n, m), $u_n \dots u_m$ represents the identity in $\pi_1(G, a)$. However, $u_n \dots u_m$ also represents $d_r \dots d_{s-1}$ where $n = i_r$, $m = i_s$, so $w_r^{-1} w_s = d_r \dots d_{s-1} = 1$ and $w_r = w_s$. Since w is doubly transient, this can occur for only finitely many pairs (n, m) with $n < 0 < m$, so there can be only finitely many pairs (n, m) with $|A(u_n \dots u_m)| = 1$ and $u_n = u_m = u_0 = a$ contradicting our assumption. □

By Corollary 12, the origin survives with positive probability in these cases. It remains to investigate the cases when G_x is not of these forms. We shall do this by eliminating colors from the percolation.

5. Eliminating colors

If x and y are two sequences and a is a color, we can consider the percolation given by the sequences x' and y' obtained by removing all occurrences of the color a from both sequences. If x is given by a Markov chain as above, then so is x'. The new transition probabilities are

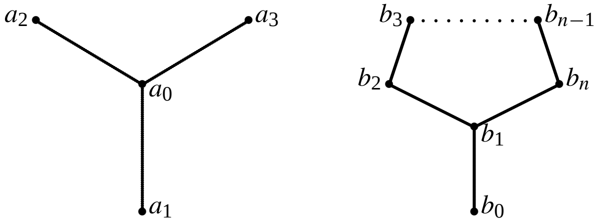
$$p'_{ij} = p_{ij} + p_{ia}p_{aj}/(1 - p_{aa}), \qquad i, j \neq a. \tag{11}$$

To be precise we need to be careful about the indices of x' relative to x. The simplest way of doing this is to condition on the event $x_0 \neq a$ and then make x'_0 correspond to x_0. The distribution of x'_0 is then the stationary distribution for the new Markov chain and the process is translation invariant.

Lemma 18. *If $x_0, y_0 \neq a$ and the origin survives in the percolation process given by x' and y' then it survives in the process given by x and y.*

Proof. Take an infinite trail in the (x', y') percolation. At each step of the trail, an edge either corresponds to an edge in the original, or crosses a row with $y_j = a$ at a point where $x_i \neq a$ or crosses a column with $x_i = a$ at a point where $y_j \neq a$. In each case, the trail can be followed in the (x, y) percolation through unblocked sites. □

Using Lemma 18 together with Corollary 17 we can deduce the survivability of percolations for almost all G_x. An *n-star* is the graph on $n+1$ vertices $a_0, \dots a_n$ with edges a_0a_i for all $1 \leq i \leq n$; the vertex a_0 is the *center* of the star. Define an *lollipop* to be the graph on $b_0, \dots, b_n$ obtained by adding a single vertex b_0 and a single edge b_0b_1 to a cycle $b_1, \dots, b_n$. We shall call an edge ab of G_x *two-way* if both p_{ab} and p_{ba} are non-zero, otherwise we shall call it *one-way*.



A 3-star. A lollipop.

Lemma 19. *If $G = G_x$ is not a path, a 3-star or a (traversable) cycle then we can remove colors as above so as to make G either have two traversable cycles or be equal to a lollipop (with its cycle traversable).*

Proof. First assume G contains a one-way edge ab with $p_{ab} = 0$ (and so $p_{ba} > 0$). Since the Markov chain is ergodic, b is accessible from a and there is a path P from a to b with all transition probabilities non-zero in the forward direction. Adding the edge ba gives a traversable cycle.

If G is a tree then it cannot have a traversable cycle and hence all the edges must be two-way. If G is not a path then there must be a vertex a with degree $d \geq 3$. Removing a gives a graph in which any two of the d neighbors of a are joined by a two-way edge, since if b, c are neigbors of a then $p'_{bc} \geq p_{ba}p_{ac} > 0$ by (11). The resulting graph has at least one cycle (since it contains K_d for some $d \geq 3$), but is not equal to a cycle unless G is a 3-star.

We can now assume G has a cycle. Either all the edges in the cycle are two-way, or there is a one-way edge. In either case G has a traversable cycle C. If C is the only cycle of G and $G \neq C$ then G contains more vertices than C. In this case we can remove each vertex of $V(G) \setminus V(C)$ in turn except one. The resulting graph G' is connected, so must contain a lollipop with traversable cycle C. If G' contains any more edges then it has two cycles otherwise G' is precisely a lollipop.

Finally, assume G has two cycles. We shall show that it has two traversable cycles. We already know that it has one traversable cycle C, say. Now let C' be another cycle. If any edges of $E(C') \setminus E(C)$ are one-way we get a traversable cycle including this edge, which is therefore not C. Otherwise all the edges of $E(C') \setminus E(C)$ are two-way. If $E(C') \cap E(C) = \emptyset$, then C' is traversable. Otherwise C' contains a path P, disjoint from C, which goes from one vertex a of C to another vertex b of C. Combining this (two-way) path, with a path from a to b in C gives a traversable cycle not equal to C. In all cases, we have at least two traversable cycles in G and the result is proved. □

We shall now deal with the case when $G = G_x$ is a lollipop. Label the vertices of the cycle of G as $1, \dots, n$ and let the remaining vertex be a and the remaining edge be $a1$. If $u = u_0 \dots u_r$ is a sequence of colors in the cycle (so $u_i \neq a$) with $u_r = u_0 = 1$, define the lifted sequence of integers $\widetilde{u}_i$ by

$$\widetilde{u}_0 = u_0 = 1, \qquad |\widetilde{u}_{i+1} - \widetilde{u}_i| \leq 1 \quad \text{and} \quad \widetilde{u}_i = u_i \bmod n. \tag{12}$$

Note that $n_u = (\widetilde{u}_r - \widetilde{u}_0)/n$ gives an integer representing the class of aua in $\pi_1(G, a) \cong \mathbb{Z}$.

Lemma 20. *Let $x = aua$ and $y = ava$ be words whose transitions are in G and whose subwords u and v do not contain a. If $|n_u| \neq |n_v|$ and $n_u, n_v \neq 0$ then any two vertices in the interior of any two sides of the rectangle of sites of $x \times y$ are connected by a traversable trail lying within this rectangle. If $n_u = 0$ and $n_v \neq 0$ then the same is true of any two vertices in the interior of the left or right hand sides of the rectangle.*

Proof. Let $u = u_0 \dots u_r$ and $v = v_0 \dots v_s$ and define $h(i, j) = \widetilde{u}_i - \widetilde{v}_j$. The site (i, j) is blocked if and only if $h(i, j) \equiv 0 \bmod n$. Also the value of $h(i, j)$ changes by at most 1 for every step in i or j. Pick any vertex in the interior of one of the sides of $x \times y$. Let C be the connected component of sites in this rectangle that can be reached from this vertex by a traversable trail within $x \times y$. If C contains any vertex from a side of $x \times y$ then it contains all vertices in the interior of this side since all these vertices are connected and open. If C does not consist of all open sites of $x \times y$, there must be a path of blocked sites crossing the rectangle. This path must start and end at two of the corners of the rectangle and go a distance at most $\sqrt{2}$ at each step. The value of $h(i, j)$ along this path is divisible by $n \geq 3$ and changes by at most two at each step. Hence, $h(i, j)$ is a constant along the path. Therefore two of the values $h(0, 0)$, $h(0, s)$, $h(r, 0)$ and $h(r, s)$ must be the same. But $h(0, 0) = 0$, $h(0, s) = nn_v$, $h(r, 0) = nn_u$ and $h(r, s) = n(n_u + n_v)$ so either $|n_u| = |n_v|$ or $n_u n_v = 0$. Finally, if $n_u = 0$, $n_v \neq 0$ then the path cannot connect the top and bottom of the rectangle, and so the left and right sides must still be connected. □

Lemma 21. *If μ_y is translation invariant, μ_x a Markov chain independent of μ_y and G_x is a lollipop, then the origin survives with positive probability.*

Proof. The sequences x and y can be written in the form $\dots au^{(-1)}au^{(0)}au^{(1)}a\dots$ and $\dots av^{(-1)}av^{(0)}av^{(1)}a\dots$ for some sequences of words $u^{(i)}$ and $v^{(j)}$ not containing a. Since we can assume by Lemma 14 that $G_y = G_x$, all $u^{(i)}$ and $v^{(j)}$ start and end at vertex 1 and have all transitions in the cycle $1, \dots, n$. Write $x'_i = \min(|n_{u^{(i)}}|, 4)$, $y'_j = \min(|n_{v^{(j)}}|, 4)$, where the colors are now considered as elements of $\{0, 1, 2, 3, 4\}$. The terms x'_i are i.i.d. random variables that can take on each of these five values with positive probability. Suppose there exists a trail P in the percolation defined by x'_i and y'_j, that passes only through sites with $x'_i \neq y'_j$ and that is also restricted to go only horizontally in or out of any site where $x'_i = 0$ and vertically in or out of any site where $y'_j = 0$. We can construct a traversable trail in the original percolation in the following manner. Each time P crosses an edge in $\mathbb{Z}^2$, pick a vertex in the interior of the corresponding side of a rectangle $au^{(i)}a \times av^{(j)}a$. By Lemma 20, these vertices can be connected by traversable trails. Finally, it follows that, with positive probability, such a P exists. Let x''_i and y''_j be the sequences x'_i and y'_j with all zeros removed. The sequence x'' is given by a Markov chain with graph K_4 (which has two traversable cycles) so we obtain a trail in the (x''_i, y''_j) process by Corollary 17. Lifting this to the (x'_i, y'_j) process in the manner of Lemma 18 gives a P with the required property. □

Theorem 22. *If μ_y is translation invariant and if μ_x is an ergodic Markov chain, independent of μ_y with graph G_x not equal to a path, an unbiased cycle or a 3-star, then the origin survives with positive probability.*

Proof. Immediate from Lemma 19, Lemma 18, Corollary 17, Corollary 12 and Lemma 21. □

Finally, we note that if $G_x = G_y$ is a 3-star with color a as the center, then the sequences x and y alternate between a and the other three colors. It is a simple exercise to see that removing the color a does not affect whether or not the origin survives with positive probability. Also removing a from x gives a Markov chain with an unbiased 3-cycle for its graph. We are now left with the cases when $G_x = G_y$ is an unbiased cycle or a path.

6. Markov chains on unbiased cycles

Assume u is a doubly infinite sequence with transitions $u_i u_{i+1}$ lying in the cycle $1, \dots, k$. Define a lifted sequence $\widetilde{u}$ of integers by

$$\widetilde{u}_0 = u_0, \qquad |\widetilde{u}_{i+1} - \widetilde{u}_i| \le 1 \quad \text{and} \quad \widetilde{u}_i \equiv u_i \bmod k. \tag{13}$$

This defines a unique doubly infinite sequence $\widetilde{u}$. For two such sequences x and y, define $h(i, j) = \widetilde{x}_i - \widetilde{y}_j$. Since $\widetilde{u} \equiv u \bmod k$, (i, j) is blocked if and only if $h(i, j)$ is divisible by k. Also, $h(i, j)$ varies by at most 1 along each step of any traversable path, so its value cannot cross a multiple of k. In particular, the points (i, j) that can be reached from the origin must have $|h(i, j)| < k$.

If u is given by a Markov chain on an unbiased cycle G_u, then $\widetilde{u}$ will look something like a symmetric random walk on $\mathbb{Z}$. Unfortunately, it is not quite such a walk since $p_{i,i+1}$ and $p_{i,i-1}$ depend on i modulo k. However, we know from Lemma 15 that the corresponding walk on $\pi_1(G)$ is a true symmetric random walk. Hence, on a sufficiently large scale $\widetilde{u}$ will look like a symmetric random walk.

Lemma 23. *If there exists constants $i_1 < 0 < i_2$, $j_1 < 0 < j_2$ with $\widetilde{x}_{i_1}, \widetilde{x}_{i_2} \ge \max_{j_1<j<j_2} \widetilde{y}_j + k$ and $\widetilde{y}_{j_1}, \widetilde{y}_{j_2} \le \min_{i_1<i<i_2} \widetilde{x}_i - k$ then the percolation does not survive.*

Proof. Note that $h(i, j) \ge k$ on all the vertices on the boundary of the rectangle with corners (i_1, j_1), (i_2, j_1), (i_2, j_2) and (i_1, j_2). None of these points can be reached from the origin, and so no traversable path can leave the rectangle. □

Corollary 24. *If μ_x and μ_y are independent Markov chains with $G_x = G_y$ an unbiased cycle $1, \dots, k$, then with probability 1 the origin does not survive.*

Proof. For such x and y, $\widetilde{x}$ and $\widetilde{y}$ will be independent random walks on $\mathbb{Z}$. We shall verify that the conditions of Lemma 23 hold. Pick any values for i_1, i_2, j_1 and j_2. Assume we are given the values of $\widetilde{x}$, $\widetilde{y}$ for $i_1 \le i \le i_2$, $j_1 \le j \le j_2$. Define

$$R = \lceil \max(\max_{i_1 \le i \le i_2} |\widetilde{x}_i|, \max_{j_1 \le i \le j_2} |\widetilde{y}_j|, k)/k \rceil. \tag{14}$$

There is a positive probability (in fact exactly $\frac{1}{5}$) of a symmetric random walk starting at 0 hitting $+4R$ before hitting $-R$, bounded below independently of R. Therefore, there is the same probability of $\widetilde{x}_i$ increasing by $4Rk$ before decreasing by Rk, since a change of k corresponds to one step in the random walk on $\pi_1(G)$. So there is a positive probability (at least 5^{-4}) bounded below independently of the values of x_i and y_j in $i_1 \le i \le i_2$ and $j_1 \le j \le j_2$ of finding $i'_1, \dots, j'_2$ with

$$\begin{aligned} &i'_1 < i_1 < i_2 < i'_2, && \widetilde{x}_{i'_1} = \widetilde{x}_{i'_2} \ge 3Rk, && \min_{i'_1<i<i'_2} \widetilde{x}_i \ge -2Rk, \\ &j'_1 < j_1 < j_2 < j'_2, && \widetilde{y}_{j'_1} = \widetilde{y}_{j'_2} \le -3Rk, && \max_{j'_1<j<j'_2} \widetilde{y}_j \le 2Rk. \end{aligned} \tag{15}$$

If these conditions hold, the conditions of Lemma 23 apply and the percolation does not survive. Otherwise start with these new values of $i_1, \dots, j_2$ and repeat the process. Since there is a probability of success at each stage, bounded away from zero, we get the result with probability 1. □

Despite the fact that the percolation does not survive, the average number of vertices that can be reached from the origin is infinite, as the following result shows.

Theorem 25. *If μ_x is a Markov chain with G_x an unbiased cycle $1, \dots, k$, then there is a $c > 0$ such that for all $N \geq 1$, the probability of at least N vertices being reached from the origin is at least $cN^{-1/2}$.*

Proof. Let $R = \lfloor \sqrt{N} \rfloor$. For all sufficiently large N, the probability that $\widetilde{x}_i < -R$ for all $-2N \leq i < -N$ and $\widetilde{x}_i > R$ for all $N < i \leq 2N$ is bounded below by some $c_1 > 0$ independently of N. Assume these conditions hold. Simple compressions on x correspond to simple compressions on $\widetilde{x}$, and $\widetilde{x}$ is compressed precisely when it is linear. By the proof of Theorem 6, each term $\widetilde{a}$ in $A(\widetilde{x}_{-2N} \dots \widetilde{x}_{2N})$ with $\widetilde{a} \not\equiv y_0 \bmod k$ gives rise to a site $(i, 0)$ that escapes the square $[-2N, 2N] \times [-2N, 2N]$ and has $\widetilde{x}_i = \widetilde{a}$. If $|\widetilde{a}| \leq R$ then $|i| \leq N$, and each such $\widetilde{a}$ occurs (precisely once) in $A(\widetilde{x}_{-2N} \dots \widetilde{x}_{2N})$. So there are at least $\lfloor \frac{k-1}{k}(2R+1) \rfloor$ sites $(i, 0)$ with $|i| \leq N$ that survive a distance at least N. Therefore the probability of any one site reaching N others is at least $c_1 \lfloor \frac{k-1}{k}(2R+1) \rfloor / (2N)$ for sufficiently large N. Hence there is a $c > 0$ such that this probability is at least $cN^{-1/2}$ for all N. □

As a contrast, we have the following result for paths.

Theorem 26. *If μ_x and μ_y are independent Markov chains with $G_x = G_y$ a path $1, \dots, k$, then the average number of sites reachable from the origin is finite.*

Proof. Without loss of generality, assume $x_0 > y_0$. Since $G_x = G_y$ is a path, $\widetilde{x} = x$ and $\widetilde{y} = y$. Pick $i_1 < 0 < i_2$ such that $x_{i_1} = x_{i_2} = 1$ and pick $j_1 < 0 < j_2$ such that $y_{j_1} = y_{j_2} = k$. For a point (i, j) on the boundary of the rectangle with corners $(i_1, j_1), \dots, (i_2, j_2)$, either $x_i = 1$ or $y_j = k$, so $x_i - y_j \leq 0$. Hence $h(i, j) \leq 0$ on the vertices of the boundary of this rectangle. But $h(0, 0) > 0$ and so $k > h(i, j) > 0$ for all sites that can be reached from the origin. The total number of sites that can be reached from the origin is therefore at most $(i_2 - i_1)(j_2 - j_1)$. But if we choose the i's and j's with smallest absolute value then $\mathbb{P}(i_i = n)$ and $\mathbb{P}(j_i = n)$ are bounded by decreasing exponentials in $|n|$ (even after conditioning on $x_0 > y_0$). A simple calculation now shows that $\mathbb{E}((i_2 - i_1)(j_2 - j_1)) < \infty$. □

Appendix A. Random walks on free groups

Let $\{d_t : t \geq 0\}$ be i.i.d. random variables in a free group G and let $\{w_t : t \geq 0\}$ be a random walk defined by $w_0 = 1_G$ and $w_{t+1} = w_t d_t$. For $g \in G$, let $p_g = \mathbb{P}(d_t = g)$. Our aim is to prove the following results.

Theorem 27. *If $G = \mathbb{Z}$, $\mathbb{E}(d_t) \neq 0$ and p_i is bounded above by a decreasing exponential in $|i|$ then there exists $\lambda < 1$ with $\mathbb{P}(w_t = 0) \leq \lambda^t$.*

Proof. The conditions on the d_t imply that the moment generating function $f(\theta) = \mathbb{E}(e^{\theta d_t})$ exists in some interval around 0 and $f'(0) \neq 0$. Hence there exists a θ with $\lambda = f(\theta) < 0$. Now

$$\mathbb{P}(w_t = 0) \leq \mathbb{E}(e^{\theta w_t}) = (f(\theta))^t = \lambda^t. \tag{16}$$

□

Theorem 28. *If $p_a, p_b > 0$ for some pair a, b of independent elements of G then there exists $\lambda < 1$ with $\mathbb{P}(w_t = 1_G) \le \lambda^t$.*

This seemingly obvious result does not appear to have been proved in the literature, so we give a proof here. This proof is based on the general approach of Jerrum and Sinclair [9] (see also [2], Chapter 9).

Define the *conductance* Φ of G by

$$\Phi = \inf_{0<|U|<\infty} \frac{1}{|U|} \sum_{g\in U, h\notin U} p_{g^{-1}h}, \tag{17}$$

where the infimum is taken over all finite non-empty subsets U of G. Since $\sum_{h\in G} p_h = 1$, it is clear that $0 \le \Phi \le 1$. We shall first prove that $\Phi > 0$. For a set U and an element $g \in G$, denote by Ug the set $\{ug : u \in U\}$.

Lemma 29. *If $p_a, p_b > 0$ for some pair a, b of independent elements of G then $\Phi > 0$.*

Proof. In the definition of Φ, restrict the sum to pairs g, h such that $g^{-1}h = a$ or b. We need to count the number of pairs (g, h) with $g \in U$, $h \notin U$ and $h = ga$ or $h = gb$. Construct a directed graph on G by setting $V(G) = G$ and $E(G)$ equal to the set of pairs (g, h) with $h = ga$ or $h = gb$. Since a and b are independent, the graph is a forest (any cycle would give a relation between a and b). The subgraph spanned by U is therefore also a forest and so the number of edges (g, h) with $g, h \in U$ is less than $|U|$ (at most one of (g, h) and (h, g) can be an edge and an undirected forest has fewer edges than vertices). On the other hand, the number of edges (g, h) with $g \in U$, $h \in G$ is precisely $2|U|$. Therefore the number of edges with $g \in U$ and $h \notin U$ is at least $|U|$. Therefore $\Phi \ge \inf_{0<|U|<\infty} \frac{1}{|U|}|U|\, p \ge p > 0$, where $p = \min(p_a, p_b)$. □

Replace the random walk w_t by a "Lazy" random walk $w'_t = w_{f(t)}$ defined by $f(0) = 0$ and $f(t+1) = f(t)$ or $f(t)+1$ with probability $\frac{1}{2}$ independently of all d_t and other increments of $f(t)$. Clearly w'_t is also a random walk with transition probabilities $p'_{1_G} = \frac{1}{2}(1 + p_{1_G})$ and $p'_g = \frac{1}{2}p_g$ for $g \ne 1_G$. It is also clear that w'_t is just a slowed down version of w_t. Let $u_g^{(t)} = \mathbb{P}(w'_t = g)$ be the distribution of w'_t at some time t and define $h(t) = \sum (u_g^{(t)})^2$. We will show that $h(t)$ decreases exponentially with t.

Lemma 30. $h(t+1) \le (1 - \frac{\Phi^2}{4})h(t)$.

Proof. Let $u_h = u_h^{(t)}$ and $u'_h = u_h^{(t+1)}$. Then

$$u'_h = \frac{1}{2}u_h + \frac{1}{2}\sum_g p_g u_{hg^{-1}} = \frac{1}{2}\sum_g p_g(u_h + u_{hg^{-1}}). \tag{18}$$

Writing $h(t+1)$ in terms of u_h and applying Cauchy-Schwarz to the inner sum gives

$$\begin{aligned}
h(t+1) &= \frac{1}{4}\sum_h \Big(\sum_g p_g(u_h + u_{hg^{-1}})\Big)^2 \\
&\le \frac{1}{4}\sum_{g,h} p_g(u_h + u_{hg^{-1}})^2 \\
&= \frac{1}{4}\sum_{g,h} p_g\big(2(u_h^2 + u_{hg^{-1}}^2) - (u_h - u_{hg^{-1}})^2\big) \\
&= h(t) - \frac{1}{4}\sum_{g,h} p_{g^{-1}h}(u_h - u_g)^2. \qquad (19)
\end{aligned}$$

In the last line we have used $\sum p_g = 1$ to get the first term and we replaced g by $g^{-1}h$ in the sum. Using Cauchy-Schwarz again, we have

$$\begin{aligned}
\Big(\sum_{g,h} p_{g^{-1}h}|u_h^2 - u_g^2|\Big)^2 &= \Big(\sum_{g,h} p_{g^{-1}h}|u_h + u_g||u_h - u_g|\Big)^2 \\
&\le \sum_{g,h} p_{g^{-1}h}(u_h + u_g)^2 \sum_{g,h} p_{g^{-1}h}(u_h - u_g)^2 \\
&\le \sum_{g,h} p_{g^{-1}h}2(u_h^2 + u_g^2) \sum_{g,h} p_{g^{-1}h}(u_h - u_g)^2 \\
&= 4h(t)\sum_{g,h} p_{g^{-1}h}(u_h - u_g)^2. \qquad (20)
\end{aligned}$$

Hence, it is enough to show

$$\sum_{g,h} p_{g^{-1}h}|u_h^2 - u_g^2| \ge 2\Phi h(t). \qquad (21)$$

Order the $g \in G$ as g_i with $u_{g_i} \ge u_{g_{i+1}}$. This is possible since $\sum u_g = 1$ and so there are only finitely many g with $u_g \ge \epsilon$ for any $\epsilon > 0$. Therefore

$$\begin{aligned}
\sum_{g,h} p_{g^{-1}h}|u_h^2 - u_g^2| &= 2\sum_{i<j} p_{g_j^{-1}g_i}(u_{g_i}^2 - u_{g_j}^2) \\
&= 2\sum_l (u_{g_l}^2 - u_{g_{l+1}}^2) \sum_{i\le l,\, j>l} p_{g_j^{-1}g_i} \\
&\ge 2\sum_l (u_{g_l}^2 - u_{g_{l+1}}^2)\, l\Phi \qquad (22) \\
&= 2\Phi \sum_l u_{g_l}^2 = 2\Phi h(t). \qquad (23)
\end{aligned}$$

We used the definition of Φ with $U = \{g_i : i \le l\}$ to obtain (22). We also used the fact that $lu_{g_{l+1}}^2 \to 0$ as $l \to \infty$ to simplify the telescoping series and get (23). □

Proof of Theorem 28.
Since $h(0) = 1$, Lemma 30 implies $h(t) \le (1 - \frac{\Phi^2}{4})^t$. However, $h(t) \ge \mathbb{P}(w'_t = 1_G)^2$ so $\mathbb{P}(w'_t = 1_G) \le (1-\frac{\Phi^2}{4})^{t/2}$. The probability that $f(2t) = t$ is $2^{-2t}\binom{2t}{t} \ge 1/(2t+1)$, so

$$\frac{1}{2t+1}\mathbb{P}(w_t = 1_G) \le \mathbb{P}(w'_{2t} = 1_G) \le \big(1 - \frac{\Phi^2}{4}\big)^t. \tag{24}$$

Since $\Phi > 0$, we have $\mathbb{P}(w_t = 1_G) \le \lambda^t$ for some $\lambda < 1$. $\square$

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