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Adversarial resilience of matchings in bipartite random graphs

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We study the problem of finding the largest matching in a random bipartite graph after an adversary deleted some edges. The bipartite graph consists of a partition class A of size n and a partition class B of size $(1+\varepsilon)n$. Each vertex in A chooses d neighbours in B uniformly and independently at random, and an adversary then deletes, for each vertex $v \in A$, at most r edges incident to v, for some fixed $r \geq 1$. Let $\varepsilon_{r,d} := \big(\frac{r+1}{r}(\log d)/\binom{d}{r+1}\big)^{1/r}$. We show that for each $\eta > 0$ and for sufficiently large (but fixed) d, if $\varepsilon \geq (1+\eta)\varepsilon_{r,d}$ then asymptotically almost surely an adversary who deletes r edges incident to each vertex in A cannot destroy all matchings of size n. On the other hand if $\varepsilon < (1-\eta)\varepsilon_{r,d}$, then asymptotically almost surely such an adversary can destroy all matchings of size n.

Keywords and phrases: Resilience, bipartite graph, matching, random graph.

1. Introduction

We are interested in the problem of finding the largest matching in a random bipartite graph after an adversary has deleted some edges. More precisely, we consider a random bipartite graph $G = (A \cup B, E)$ with $|A| = n$ and $|B| = (1+\varepsilon)n$. Each vertex in A is adjacent to d neighbours chosen uniformly and independently at random from B. We allow repetition, so this is a multigraph. An adversary is then able to remove at most r edges adjacent to each vertex of A, with the aim of minimising the size of the largest matching.

Finding matchings in graphs is a well-studied problem and polynomial time algorithms are known to find maximum matchings, see for example [6, 13]. Recently in [12], Liu, Slotine, and Barabási used a characterisation by Lin [11] of structural controllability to show how large matchings in bipartite graphs play a crucial role to obtain bounds on the number of nodes needed to control directed networks.

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They also estimated the fraction of edges in a matching drawn randomly from several classes of graphs in a non-rigorous way using the cavity method. Rigorous proofs for most of these results can be found in [3] and also in [14]. In particular the authors determine the size of a maximum matching in a random bipartite graph with a fixed degree distribution under some mild assumption on these distributions. The application in controllability motivated us to look at a model where an adversary is present.

Melsted and Frieze [7] analysed the Karp-Sipser algorithm for the random bipartite graph where each vertex in a partition class of size n chooses d neighbours uniformly at random from a partition class of size $\tilde{n} = \alpha n$ for some $\alpha > 0$. We consider this model with an added adversary, to analyse the local resilience of the matchings in this model.

The resilience of a graph property refers to the difficulty in eliminating it by removing edges from the graph, either randomly or according to some defined process. Here we consider worst case, or adversarial resilience, in which we allow an adversary the ability to delete edges at will, with the aim of disrupting the desired property in the graph. This models both the ability of an attacker to disrupt a network, and also the worst possible case that could occur randomly.

In general allowing an adversary entirely free reign in choosing which edges to destroy provides too much power, and almost any global property can be disrupted, as for example, isolating a single vertex of low degree must automatically eliminate the possibility of a complete matching. Limiting the abilities of an adversary with respect to local conditions in targeting global properties provides more mathematically interesting results.

In analysing our particular class of bipartite graph, we are interested in a variant of local resilience, in that we allow the adversary to eliminate at most r edges incident to each vertex in one of the graph partitions, but we provide no restriction on the edges removed with respect to the neighbourhoods of the vertices in the other partition.

An excellent overview of local resilience and various results in this area can be found in the article "Local Resilience of Graphs" by Sudakov and Vu [15]. This paper provided a more systematic approach to studying resilience and ignited interest in the topic, resulting in a large number of recent results (see [1], [2], [4], [5] and [10] for a range of examples).

We will show the following theorem, where we use the notation *asymptotically almost surely (a.a.s.)* which means with probability tending to 1 as n tends to infinity.

Theorem 1.1. *Let $G = (A \cup B, E)$ with $|A| = n$ and $|B| = (1 + \varepsilon)n$ be a random bipartite (multi-)graph in which each vertex in A chooses d vertices* *uniformly and independently at random with repetition from B. For each $r \geq 1$ and each $\eta > 0$ there exists a $d_0 = d_0(r, \eta)$ such that for all $d \geq d_0$, if*

$$\varepsilon > (1+\eta)\left(\frac{(r+1)\log d}{r\binom{d}{r+1}}\right)^{1/r},$$

then asymptotically almost surely an adversary who deletes at most r edges incident to each vertex in A cannot destroy all matchings of size n. On the other hand if

$$\varepsilon < (1-\eta)\left(\frac{(r+1)\log d}{r\binom{d}{r+1}}\right)^{1/r},$$

then asymptotically almost surely such an adversary can destroy all matchings of size n.

For ease of notation, define

$$\varepsilon_{r,d} := \left(\frac{(r+1)\log d}{r\binom{d}{r+1}}\right)^{1/r}.$$

The problem of finding such resilient matchings is closely related to finding a maximum r-independent set in a random d-uniform hypergraph $\mathcal{H}^d_{\tilde{n},n}$ on $\tilde{n} \geq n$ vertices and n edges. An r-independent set in a d-uniform hypergraph is a set of vertices such that each hyperedge contains at most r vertices of this set, see for example [9]. Here, the partition class B of the bipartite graph G corresponds to the vertices of the hypergraph and the neighbourhoods of the vertices of A correspond to the edges. Clearly, an adversary can isolate all vertices corresponding to an r-independent set, and hence if the maximum size of an independent set is β then the adversary can force the size of a maximum matching to be at most $\tilde{n} - \beta$. We will use a result by Krivelevich and Sudakov [9] that implies that $\mathcal{H}^d_{\tilde{n},n}$ a.a.s. contains an independent set of size $(1-\eta)\varepsilon_{r,d}n$ where η can be chosen arbitrarily small if d is sufficiently large.

As a note on notation, asymptotically, we are interested in the probabilistic results as $n \to \infty$, and so by $o(1)$ we mean a function that tends to 0 as $n \to \infty$. But since we are also considering $d \to \infty$ (and at the same time $\varepsilon \to 0$), we may also require the use of little o notation to denote the size of terms which do not depend on n, in which case we will label them $o_d(f(d))$ to indicate that the asymptotics depend on d rather than n. As an example $\varepsilon = o_d(1)$, but neither ε nor d depend on n and hence are asymptotically constant in terms of n.

2. Upper bounds

We begin by proving the upper bound on the threshold for ε. To prove this we consider Hall's theorem.

Theorem 2.1 (Hall [8]). *For a bipartite graph G with partitions X and Y, a matching in G of size $|X|$ exists if and only if,*

$$\forall X' \subseteq X, |\Gamma(X')| \geq |X'|, \tag{1}$$

where $\Gamma(X')$ is the set of neighbours of X' in Y.

If there exists a set $X' \subseteq X$ that does not satisfy (1), we call X' a *witness* for violating Hall's condition.

We aim to give a bound on the probability of the adversary being able to restrict the size of the maximal matching. We do this by considering the probability of the existence of a set that, after deletion of edges by the adversary, could become a witness.

Theorem 2.2. *Let G be the random bipartite graph with partition sets A and B of size n and $\tilde{n} = (1+\varepsilon)n$ respectively, and each vertex of A chooses d vertices uniformly and independently at random with repetition from B. An adversary deletes at most r edges incident to each vertex of A to obtain G'. For each $\eta > 0$ there exists a $d_0 = d_0(r, \eta)$ such that for all $d \geq d_0$ and $\varepsilon > (1+\eta)\varepsilon_{r,d}$, a matching of size n still exists in G' with probability tending to 1 as $n \to \infty$.*

Proof. Fix $\eta > 0$ and assume that $\varepsilon > (1+\eta)\varepsilon_{r,d}$. Suppose that a matching of size n does not exist in G'. By Hall's Theorem at least one witness of Hall's condition exists. Consider a smallest such witness S, of size s, say. Its neighbourhood in G' must be of size $s-1$, or we could delete an element of S and still have a witness of smaller size.

For two sets $S \subseteq A$ and $S' \subseteq B$ of sizes s and $s-1$ respectively, if they form a witness in G', then for each vertex of S in G, at most r edges meet $B \setminus S'$ (which the adversary then deletes).

The probability of a given edge incident to a vertex of S also being incident to a vertex in S' is $p := \frac{s-1}{\tilde{n}}$. Let $q := 1 - p$. Then the probability that the d edges incident to a vertex of S satisfy this condition is

$$\rho := \mathbb{P}(\mathrm{Bin}(d, q) \leq r) = \sum_{i=0}^{r} \binom{d}{i} p^{d-i} q^i.$$

Note that

$$\rho \leq \binom{d}{r} p^{d-r}. \tag{2}$$

as we can bound ρ by the sum over all subsets of edges of size $d-r$ of the probability that these edges are incident to the set S.

The expected number e_s of witnesses of size s is therefore bounded by

$$e_s \leq \binom{n}{s} \binom{\tilde{n}}{s-1} \rho^s. \tag{3}$$

(The inequality may be strict as the right hand side counts witnesses with $|\Gamma(S)| < s-1$ with high multiplicity.) We split the analysis into several cases depending on s, or equivalently p. For s not too large, we use the inequality $\binom{n}{r} \leq \left(\frac{en}{r}\right)^r$ to obtain

$$\binom{n}{s}\binom{\tilde{n}}{s-1} \leq \left(\frac{en}{s}\right)^s \left(\frac{e\tilde{n}}{s-1}\right)^{s-1} \leq \left(\frac{e\tilde{n}}{s-1}\right)^{2s} = \left(\frac{e}{p}\right)^{2s}.$$

Thus

$$e_s \leq \left(\frac{e^2\rho}{p^2}\right)^s \leq \left(e^2\binom{d}{r}p^{d-r-2}\right)^s. \tag{4}$$

Case $s \leq \sqrt{n}$.

In this case $p \leq 1/\sqrt{n}$, so for $d > r+2$, $e^2\binom{d}{r}p^{d-r-2} \leq c/\sqrt{n}$ for some constant c independent of n. Hence

$$\sum_{s=1}^{\sqrt{n}} e_s \leq \frac{c/\sqrt{n}}{1-c/\sqrt{n}} = o(1).$$

Case $s \geq \sqrt{n}$ and $q \geq 2r(\log d)/(d-r-2)$.

Note that the lower bound on q implies an upper bound on s, say $s \leq s_0$. In this case

$$e^2\binom{d}{r}p^{d-r-2} = e^2\binom{d}{r}(1-q)^{d-r-2} \leq e^2\binom{d}{r}e^{-q(d-r-2)} \leq e^2 d^r d^{-2r},$$

which is less than $1/2$ for large enough d. Thus

$$\sum_{s=\lceil\sqrt{n}\rceil}^{s_0} e_s \leq \sum_{s=\lceil\sqrt{n}\rceil}^{\infty} \frac{1}{2^s} = 2^{-\lceil\sqrt{n}\rceil+1} = o(1).$$

Case $c_1/d \le q \le 2r(\log d)/(d-r-2)$ **for some sufficiently large constant** c_1 **depending only on** r**.**

As before the bounds on q imply bounds on s and we denote the corresponding bounds on s by $s_0 \le s \le s_1$. Note that $\rho = \mathbb{P}(\mathrm{Bin}(d,q) \le r)$ is increasing as q decreases. Thus, by Chebyshev's inequality,

$$\rho \le \mathbb{P}(\mathrm{Bin}(d, c_1/d) \le r) \le \frac{1}{20}$$

if c_1 is sufficiently large compared to r. For sufficiently large d we also have that $q < 0.1$, so $e^2\rho/p^2 \le 1/2$, and so by (4)

$$\sum_{s=s_0}^{s_1} e_s \le \sum_{s=s_0}^{s_1} \left(\frac{e^2\rho}{p^2}\right)^s \le \sum_{s=s_0}^{\infty} \frac{1}{2^s} = o(1)$$

as in the previous case.

Case $\left(c_2(\log d)/d^{r+1}\right)^{1/r} \le q \le c_1/d$ **where** c_2 **is a sufficiently large constant depending on** c_1 **and** r**.**

We denote the corresponding bounds on s by $s_1 \le s \le s_2$. From now on we will use the fact that

$$\begin{aligned} \rho = \mathbb{P}(\mathrm{Bin}(d,q) \le r) &\le 1 - \binom{d}{r+1} q^{r+1} p^{d-r-1} \\ &\le \exp\left(-\binom{d}{r+1} q^{r+1} p^{d-r-1}\right). \end{aligned} \tag{5}$$

Note that $s-1 = p\tilde{n} = (1-q)\tilde{n}$ and, as $s > n/2$, in this case $\binom{n}{s} \le \binom{\tilde{n}}{s-1} = \binom{\tilde{n}}{q\tilde{n}}$. Thus

$$\begin{aligned} e_s &\le \binom{\tilde{n}}{q\tilde{n}}^2 \exp\left(-\binom{d}{r+1} q^{r+1} p^{d-r-1}\right)^s \\ &\le \left(\frac{e}{q}\right)^{2q\tilde{n}} \exp\left(-\binom{d}{r+1} q^{r+1} p^{d-r-1} s\right) \\ &= \exp\left(-\binom{d}{r+1} q^{r+1} p^{d-r-1} s + 2q\tilde{n}\log(e/q)\right). \end{aligned}$$

Since $s \ge s-1 = p\tilde{n}$ we deduce that

$$\sum_{s=s_1}^{s_2} e_s \le n \exp\left(-\left(\binom{d}{r+1} q^{r+1} p^{d-r} - 2q\log(e/q)\right)\tilde{n}\right).$$

Thus it suffices to show that $\binom{d}{r+1}q^{r+1}p^{d-r} - 2q\log(e/q)$ is bounded away from 0 independently of n. But this is clear as $p^{d-r} \geq (1-c_1/d)^d \geq c_3$ for some constant $c_3 > 0$ depending on c_1, $\log(e/q) \leq 2\log d$, and $q^r \geq c_2(\log d)/d^{r+1}$.

Case $q \leq \left(c_2(\log d)/d^{r+1}\right)^{1/r}$ and $s \leq (1-ed^{-(r+1)/r})n$. In this case we have that $p^{d-1} = 1 - o_d(1)$. Note that

$$\tfrac{n-s}{n} = 1 - \tfrac{s}{n} \leq 1 - \tfrac{s-1}{n} = 1 - p(1+\varepsilon) = q - p\varepsilon$$

so, in particular, $q \geq p\varepsilon$. Also $q = 1-p \geq 1 - n/\tilde{n} = \varepsilon/(1+\varepsilon) \geq ed^{-(r+1)/r}$. Hence by (3)

$$\begin{aligned}
e_s &\leq \binom{n}{n-s}\binom{\tilde{n}}{q\tilde{n}}\exp\left(-\binom{d}{r+1}q^{r+1}p^{d-r-1}\right)^s \\
&\leq \left(\frac{en}{n-s}\right)^{n-s}\left(\frac{e}{q}\right)^{q\tilde{n}}\exp\left(-\binom{d}{r+1}q^{r+1}p^{d-r-1}s\right) \\
&\leq \left(d^{(r+1)/r}\right)^{n(q-p\varepsilon)}\left(d^{(r+1)/r}\right)^{q\tilde{n}}\exp\left(-\binom{d}{r+1}q^{r+1}p^{d-r}\tilde{n}\right) \\
&\leq \exp\left(-\left(\binom{d}{r+1}q^{r+1}p^{d-r} - \frac{r+1}{r}(2q-p\varepsilon)\log d\right)\tilde{n}\right).
\end{aligned}$$

Thus it suffices to show that $\binom{d}{r+1}q^{r+1}p^{d-r} - \frac{r+1}{r}(2q-p\varepsilon)\log d$ is bounded away from 0 independently of n. But $(q-p\varepsilon)^2 \geq 0$, so $2q - p\varepsilon \geq q^2/p\varepsilon$ and, as we noted above, $q \geq p\varepsilon$, so $(q/p\varepsilon)^{r-1} \geq 1$. Thus

$$\begin{aligned}
\frac{r+1}{r}(2q-p\varepsilon)\log d &\leq \frac{r+1}{r}\cdot\frac{q^2}{p\varepsilon}\log d \\
&\leq \frac{r+1}{r}\cdot\frac{q^{r+1}}{p^r\varepsilon^r}\log d \\
&\leq \binom{d}{r+1}\frac{q^{r+1}}{p^r}(1+\eta)^{-r} \\
&\leq \binom{d}{r+1}q^{r+1}p^{d-r}\left(1-\frac{\eta}{2}\right)
\end{aligned}$$

for sufficiently large d.

Case $q \leq \left(c_2\log d/d^{r+1}\right)^{1/r}$ and $s \geq (1-ed^{-(r+1)/r})n$.

In this case we have $\binom{n}{n-s} \leq \binom{n}{end^{-(r+1)/r}}$. As above we have

$$e_s \leq (d^{(r+1)/r})^{end^{(r+1)/r}}(d^{(r+1)/r})^{q\tilde{n}}\exp\left(-\binom{d}{r+1}q^{r+1}p^{d-r}\tilde{n}\right)$$

$$\leq \exp\left(-\left(\binom{d}{r+1}q^{r+1}p^{d-r} - \frac{r+1}{r}\left(q + ed^{-(r+1)/r}\right)\log d\right)\tilde{n}\right).$$

Now $q \geq \varepsilon/(1+\varepsilon)$ so

$$\frac{r+1}{r}\left(q + ed^{-(r+1)/r}\right)\log d \leq \frac{r+1}{r}q\left(1+\frac{\eta}{6}\right)\log d$$

while

$$\binom{d}{r+1}q^{r+1}p^{d-r} \geq \binom{d}{r+1}\frac{q\varepsilon^r p^{d-r}}{(1+\varepsilon)^r} \geq \frac{r+1}{r}q\left(1+\frac{\eta}{3}\right)\log d.$$

Thus $e_s \leq \exp(-(\frac{r+1}{r}q\log d)\tilde{n}\eta/6) = o(1/n)$ and the proof is complete. □

3. Lower bound

Instead of looking at a matching in a bipartite graph we consider the random d-uniform hypergraph $\mathcal{H}^d_{\tilde{n},n}$ consisting of $\tilde{n} = (1+\varepsilon)n$ vertices (although we in fact, prove the required hypergraph result for all $\tilde{n} \geq n$) and n distinct hyperedges. The correspondence between these two models is fairly simple. For each of the vertices of A, its neighbourhood consists of d vertices in B, chosen uniformly and independently at random. If there are no multiple edges then this results in a uniformly chosen random subset of B of size d, equivalent to the choice of edges in the random hypergraph $\mathcal{H}^d_{\tilde{n},n}$. More generally, for each vertex $v \in A$, choose independently and uniformly at random a d-set that contains the neighbourhood $\Gamma(v)$. If there are no multiple edges incident with v then there is only one choice of such a d-set, namely $\Gamma(v)$, but even in general the d-set is uniformly distributed over all d-sets of B, independently for each vertex $v \in A$. Thus we obtain a hypergraph with the same distribution as $\mathcal{H}^d_{\tilde{n},n}$. We define the random subset $M \subseteq B$ as the set of all vertices of B that are incident to at least one multiple edge.

We aim to find an r-independent set in $\mathcal{H}^d_{\tilde{n},n}$, that is, a set I of vertices such that each hyperedge intersects with at most r vertices in I. Clearly, given an r-independent set $I \subseteq B$, the adversary can isolate the vertices of $I \setminus M$ in the set B. The adversary can go through the vertices of $I \setminus M$, eliminating their incident edges and never need to remove r edges incident to a single vertex in A. Hence if the maximum r independent set is of size bigger than $\varepsilon n + |M|$ then the adversary can ensure that a matching of size n does not exist. We first show that M is small.

Lemma 3.1. $\mathbb{E}(|M|) \leq \binom{d}{2}$, *independently of* n.

Proof. The size of M is clearly bounded by the total number of pairs of parallel edges. Hence

$$\mathbb{E}(|M|) \le n\binom{d}{2}\frac{1}{\tilde{n}} \le \binom{d}{2}. \qquad \square$$

By Markov's inequality, a.a.s. $|M| \le \sqrt{n}$, say, as $n \to \infty$. Thus the lower bound will follow from the following theorem.

Theorem 3.2. *For each $r \ge 1$ and each $\eta > 0$ there exists a $d_0 = d_0(r, \eta)$ such that for all $d \ge d_0$, there exists an r-independent set in the random d-uniform hypergraph $\mathcal{H}^d_{\tilde{n},n}$ consisting of $\tilde{n} \ge n$ vertices and n edges, which asymptotically almost surely is at least of size*

$$(1 - \eta)\varepsilon_{r,d}\, n.$$

Proof. We begin by noting that if we prove the result for $\tilde{n} = n$ then the result holds for all $\tilde{n} > n$. This follows from noting that if $\tilde{n} > n$ then we can arbitrarily discard $\tilde{n} - n$ vertices and consider the hypergraph induced on the remaining n vertices. For any hyperedge entirely contained within these n vertices, we keep it. For a hyperedge that contained $1 \le d' \le d$ vertices from the set of discarded vertices, we choose d' new vertices from the remaining n vertices, ensuring we do not create any multiple edges or add the same vertex twice to a single edge. This new hypergraph on n vertices, has n hyperedges, chosen uniformly at random from all possible sets of size d of these vertices. This is equivalent to $\mathcal{H}^d_{n,n}$.

The process that generates this auxiliary hypergraph, can, at worst, decrease the size of the maximum r-independent set. This is because any r-independent set in the new auxiliary hypergraph must also be one for the original $\mathcal{H}^d_{\tilde{n},n}$ hypergraph, as the n vertices used for the auxiliary are only removed from hyperedges in reverting to the original hypergraph. Therefore, if we prove the theorem for $\tilde{n} = n$, then this auxiliary hypergraph must contain an r-independent set of the required size, which we note does not depend on $\tilde{n}$, and as such the original hypergraph must contain such a set too.

We actually use a stronger result for the random d-uniform hypergraph $\mathcal{H}^d_{n,p}$ which has n vertices and each subset of the vertices of size d is a hyperedge with probability p independently of the presence or absence of all other hyperedges. Here p is such that the expected number of hyperedges equals n, that is,

$$\binom{n}{d}p = n. \tag{6}$$

This is indeed a stronger result. For example, suppose the maximum r-independent set is a.a.s. of size at least k in $\mathcal{H}^d_{n,p}$. As this hypergraph has at least n edges with probability bounded away from zero, the maximum size of an r-independent set is a.a.s. at least k, even conditioned on the hypergraph having at least n hyperedges. But then a.a.s. there is an r-independent set of size at least k in the hypergraph obtained from this graph by selecting at random n hyperedges from this graph, which is distributed precisely as $\mathcal{H}^d_{n,n}$.

We use the following result by Krivelevich and Sudakov.

Theorem 3.3 ([9])**.** *For every $1 \leq r \leq d-1$ there exists a constant γ_0 such that if*

$$\gamma = \gamma(n,p) = r\binom{d-1}{r}\binom{n-1}{d-1}p \geq \gamma_0$$

and $\gamma = o(n^r)$ then a.a.s. there exists an r-independent set of size at least

$$\left(\frac{\gamma}{(r+1)\log\gamma}\left(1+\frac{1}{\log^{0.1}\gamma}\right)\right)^{-\frac{1}{r}} n$$

in a d-uniform hypergraph $\mathcal{H}^d_{n,p}$.

We have $p = n/\binom{n}{d}$ and hence

$$\gamma = r\binom{d-1}{r}d = \frac{d!}{(d-r-1)!(r-1)!} = \binom{d}{r+1}(r+1)r.$$

Thus the conditions of the Theorem 3.3 are satisfied and we have for sufficiently large d that

$$\left(1+\frac{1}{\log^{0.1}\gamma}\right)^{-\frac{1}{r}} \geq 1-\frac{\eta}{2}$$

and

$$\left(\frac{(r+1)\log\gamma}{\gamma}\right)^{\frac{1}{r}} \geq \left(\frac{\log\gamma}{r\binom{d}{r+1}}\right)^{\frac{1}{r}} \geq \left(1-\frac{\eta}{2}\right)\left(\frac{(r+1)\log d}{r\binom{d}{r+1}}\right)^{\frac{1}{r}}. \qquad \square$$

4. Conclusions/open problems

Although we have proven a threshold for the lower and upper bounds which are asymptotically equal as $d \to \infty$, it seems likely, that a threshold should exist for each d. In other words, we conjecture that there exists constants $c_{r,d}$ such that for all $1 \leq r \leq d-2$ and $\eta > 0$, if $\varepsilon > c_{r,d} + \eta$ then a.a.s. a matching of size n can be found, while for $\varepsilon < c_{r,d} - \eta$ there is a.a.s. a strategy for the adversary that reduces the size of the maximal matching below n. If this conjecture is true then we know that for each fixed r, $c_{r,d} = (1 + o_d(1))\left(\frac{r+1}{r}(\log d)/\binom{d}{r+1}\right)^{1/r}$ as d tends to infinity. Note that this conjecture fails for $d < r+2$. Indeed, the adversary can simply delete a random choice of $d-1 \leq r$ edges from each vertex in A and then in the resulting graph we a.a.s. have two vertices in A with the same remaining neighbour in B. On the other hand it is not hard to see that Theorem 2.2 can be strengthened so as to give a finite bound on ε even for $d = r+2$. The proof required d to be large at several points, however, following the strategy of the proof, it is easy to show that $d \geq r+2$ is enough to get a finite bound. For example, if $\tilde{n} \geq 10\binom{d}{r}n$ then, recalling that e_s is the expected number of witnesses of size s and $p := \frac{s-1}{\tilde{n}}$, we have,

$$
\begin{aligned}
e_s &\leq \binom{n}{s}\binom{\tilde{n}}{s-1}\left(\binom{d}{r}p^{d-r}\right)^s \\
&\leq \left(\frac{en}{s}\right)^s \left(\frac{e\tilde{n}}{s-1}\right)^{s-1} \left(\binom{d}{r}p^2\right)^s \\
&\leq \left(\frac{e}{10\binom{d}{r}p}\right)^s \left(\frac{e}{p}\right)^{s-1} \left(\binom{d}{r}p^2\right)^s \\
&\leq (p/e)(e^2/10)^s \\
&\leq (s/ne)(e^2/10)^s.
\end{aligned}
$$

Noting that

$$\sum_{s=1}^{n} (s/ne)(e^2/10)^s = O\left(\frac{1}{n}\right),$$

it is clear that we have $\sum_{s\geq 1} e_s = o(1)$ as required.

Although we have identified the threshold for which an adversary can and cannot destroy the complete matching when subject to the restriction of removing at most r edges incident to each vertex of A, there remains a number of interesting problems that would arise from allowing the adversary greater or differing powers in modifying G. The case where the adversary is able to delete n edges globally allows the adversary to easily isolate a linear proportion of the vertices, while equally, a matching still exists that covers a linear proportion of the vertices. In both cases a simple greedy algorithm provides fairly simple bounds, but finding the exact size of the largest remaining matching seems challenging and certainly would require further insight in tackling.

Another problem to consider would be the case of analysing the size of the maximum matching for values of r, d and ε for which we have shown that the adversary can eliminate a matching of size n. The use of our graph model was motivated by its use in [7] which analysed the size of the maximum matching in the same model but without an adversary removing edges, for all values of d, and it would be interesting to know what the behaviour of the size of the maximum matching becomes for small values of d once an adversary is introduced.

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