Department of Mathematical Sciences
PhD Qualifying Exam

Name: ________________________________

Examination:	**Real Variables I & II**		
Date of Exam:	08/15/2022	**Duration:**	180 Minutes
No. pages:	12	**No. problems:**	8

Instructions:

This exam consists of 8 problems, grouped by semester: Problems 1-4 cover material from the first semester, while Problems 5-8 are taken from the second semester. However, you may use techniques from either semester to work on the problems. When you are asked to prove a known result, make sure you provide the key arguments needed for the proof.

Please observe the following rules when choosing the problems you intend to work on:

- **Solve at least two problems from among Problems 1-4.**
- **Solve at least two problems from among Problems 5-8.**
- **Solve a total of at least 5 problems.** Hence, you may choose your 5th problem freely.
- **Provide clear and concise justifications for your answers.**

Note: Completeness of solutions is an important factor in earning a passing grade.
Write your solutions in the space provided. If you need more space, three additional, empty pages are attached at the end of the exam. You have 3 hours to finish your work. Good luck!

Please do not write below this line.

Problem	1	2	3	4	5	6	7	8	Total
Score									
Complete									

	Pass	Fail
MS		
PhD		

Problem 1:

Denote the Lebesgue measure on $\mathbb{R}$ by m.

(a) State the Monotone Convergence Theorem and the Dominated Convergence Theorem of Lebesgue Integration.

(b) Let f be a measurable function on $[0,1]$. Prove that the limit $\lim_n \int_{[0,1]} |f|^n \, dm$ either equals ∞ or $m\left(\left\{x \in [0,1] \,\middle|\, |f(x)| = 1\right\}\right)$.

Problem 2:

Denote the Lebesgue measure on $\mathbb{R}$ by m.

(a) Suppose $E \subset \mathbb{R}$ is measurable such that $m(E) < \infty$. Prove that for a.e. $x \in \mathbb{R}$,

$$\lim_{\substack{h \to 0 \\ h > 0}} \frac{m(E \cap [x, x+h))}{h} = \chi_E(x).$$

Note: χ_E denotes the characteristic function (or indicator function) of the set E.

(b) Suppose $g \in L^p(\mathbb{R})$ for some $1 \leq p < \infty$. Prove that $\lim_n \int_{[n,n+1]} g \, dm = 0$.

Is the result also true for $p = \infty$?

Problem 3:

Suppose (f_n) is a Cauchy sequence in $L^p(\mathbb{R})$ for some $1 \leq p < \infty$.

(a) Give an example which shows that the sequence (f_n) may not converge pointwise a.e.

(b) Prove: There exists a subsequence (f_{n_k}) which converges pointwise a.e.

Problem 4:

Denote the Lebesgue measure on $\mathbb{R}$ by m.

(a) State and prove Jensen's Inequality for integrable functions on $[0,1]$.

(b) Let f be an integrable function on $[0,1]$ such that $1+f>0$ a.e. and $\int_{[0,1]} f\,dm = 1$. Prove:

$$\int_{[0,1]} f\,\ln(1+f)\,dm \geq \ln 2.$$

Problem 5:

(a) Let $\mathcal{H}$ be a Hilbert space. Show that every nonempty, closed, convex set $C \subset \mathcal{H}$ contains a unique element of smallest norm.

(b) Give an example which shows that the conclusion of part (a) fails if the set C is not convex.

(c) Give an example which shows that the conclusion of part (a) fails if the set C is not closed.

Problem 6:

Let $\mathcal{X}$ be a Banach space.

(a) Suppose ϕ is a bounded linear functional on $\mathcal{X}$ such that $\phi \neq 0$. Show that the kernel of ϕ, $\ker \phi$, is a proper, closed subspace of $\mathcal{X}$. Conclude that $\ker \phi$ is nowhere dense.

(b) Let (ϕ_n) be a sequence of bounded linear functionals on $\mathcal{X}$ such that $\phi_n \neq 0$ for all $n \in \mathbb{N}$. Prove: There exists $x \in \mathcal{X}$ such that $\phi_n(x) \neq 0$ for all $n \in \mathbb{N}$.

Problem 7:

Let $T : l^2 \to l^2$ be the bounded linear operator, given by $T(\,(x_n)\,) = \left(\frac{1}{n}\, x_n\right)$ for $(x_n) \in l^2$.

(a) Prove: T maps the closed unit ball in l^2 into a compact subset of l^2.

(b) Prove or disprove: T is open.

Problem 8:

(a) State the Radon-Nikodym Theorem.

(b) Let $(X, \mathcal{A})$ be a measurable space with σ-finite measures λ and μ. Suppose that $\lambda \ll \mu$ and $\mu \ll \lambda$. Prove: There exists a nonnegative, measurable function h on X such that

- $\lambda(A) = \int_A h \, d\mu$ for all $A \in \mathcal{A}$,
- $h > 0$ μ-a.e. and λ-a.e.,
- $\mu(A) = \int_A h^{-1} \, d\lambda$ for all $A \in \mathcal{A}$.

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