Statistics Ph.D. Qualifying Exam: Part II

October 25, 2024

Student Name: ______________________________

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1. Answer 8 out of 12 problems. Mark the problems you selected in the following table.

Problem	1	2	3	4	5	6	7	8	9	10	11	12
Selected												
Scores												

2. Write your answer right after each problem selected, attach more pages if necessary. Assemble your work in right order.

3. You can use the N(0,1) distribution table as attached.

1. Suppose $X \sim \text{Bernoulli}(1/2)$ and $Y \sim \text{Bernoulli}(p)$ where $p \in (0, 1)$ is an unknown parameter. Suppose X and Y are independent. Due to aggregation, we do not get to observe X or Y. Instead, we only get to observe $Z = X + Y$. Suppose Z_1, Z_2, ..., Z_n is an i.i.d. sample from $f_Z(z)$.

   (a) Find the probability mass function of X.

   (b) Give conditions for which the MLE of p exists and determine what the MLE is when these conditions are satisfied.

   (c) Show the MLE is an unbiased estimator.

2. Suppose that $X_1, X_2, ..., X_n$ is an independent and identically distributed (iid) sample from the uniform distribution $U(-\theta, \theta)$ with an unknown parameter $\theta > 0$. Let $X_{(1)} = \min(X_1, X_2, ..., X_n)$ and $X_{(n)} = \max(X_1, X_2, ..., X_n)$ be the sample minimum and sample maximum respectively.

   (a) Prove that $T = (X_{(1)}, X_{(n)})$ is a sufficient statistic for θ.

   (b) Is T minimal sufficient for θ? If so, prove it; if not, find a minimal sufficient statistic.

   (c) Find the maximum likelihood estimate (MLE), $\hat{\theta}$, of θ based on $X_1, X_2, ..., X_n$.

3. Let $Z \sim Beta(.5 + \theta/\pi, .5 - \theta/\pi)$ for $\theta \in (-\pi/2, \pi/2)$. Define the transformed variable

$$Y = \frac{1}{\pi} log(\frac{Z}{1-Z}).$$

Show that the density of Y is proportional to

$$f(y) \propto \frac{e^{y\theta}}{2\cosh(y\pi/2)},$$

where $\cosh(y) = (e^{-y} + e^{y})/2$. Recall that the density for a $Beta(\alpha, \beta)$ is proportional to $z^{\alpha-1}(1-z)^{\beta-1}$ where $z \in (0, 1)$.

4. Let (X_1, Y_1), ..., (X_n, Y_n) be independent and identically distributed sample from a bivariate normal distribution with mean (μ, η) and covariance matrix $\Sigma = diag(\sigma^2, \tau^2)$.

   (a) If $\mu, \eta, \sigma^2, \tau^2$ are unknown parameters, find the sufficient and complete statistic for $(\mu, \eta, \sigma^2, \tau^2)$.

   (b) Find the maximum likelihood estimate and uniformly minimum variance unbiased estimate (UMVUE) for μ, η, σ^2, and τ^2

5. Let $X_1, X_2, \ldots, X_n$ be a random sample from a distribution whose pdf is

$$f(x;\theta) = \frac{\log(\theta)}{\theta - 1}\theta^x, \quad 0 < x < 1,\, \theta > 1.$$

Is there a function of θ, say $g(\theta)$, for which there exists an unbiased estimator whose variance attains the Cramér-Rao lower bound? If so, find it. If not, show why not.

6. Let $\mathbf{X} = (X_1, ..., X_n)$ be a random sample from the Poisson distribution Poisson(λ). Let λ have Gamma(a, b) distribution

$$p(x|a,b) = \frac{x^{a-1}e^{-x/b}}{b^a\Gamma(a)}, \quad x > 0$$

where $a = 4$ and $b = 2$.

(a) Find the posterior distribution of λ given data $\mathbf{X}$.

(b) Construct a Bayesian test for $H_0 : \lambda \geq 2$ versus $H_1 : \lambda < 2$. What is the rejection region of this test? Assume that H_0 and H_1 are equally important.

7. Let $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_m$ be two independent random samples from beta$(\mu, 1)$ and beta$(\theta, 1)$ populations, respectively.

(a) Find a likelihood ratio test of

$$H_0 : \mu = \theta \quad \text{versus} \quad H_1 : \mu \neq \theta.$$

(b) Show that the test in part (a) can be based on the statistic

$$T = \frac{\sum_{i=1}^{n} \log X_i}{\sum_{i=1}^{n} \log X_i + \sum_{j=1}^{m} \log Y_j}.$$

8. Let $X_1, ..., X_n$ be a random sample from the Gamma distribution with the pdf

$$p(x|b) = \frac{x^{a-1}e^{-x/b}}{b^a\Gamma(a)}, \quad x > 0$$

where $a > 0$ is known and $b > 0$ is unknown.

(a) Find the sufficient statistic for b.

(b) Find the conditional distribution of X_1 given the sufficient statistic.

(c) Find the UMVUE of $P(2 < X < 4)$.

9. Suppose that X_1, X_2 and X_3 are independently distributed of each other. For $i = 1, 2, 3$, let the probability density function X_i be

$$f_i(x) = \begin{cases} \lambda_i e^{-\lambda_i x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

(a) Find $P(\min(X_1, X_2) < X_3)$ and $P(X_1 < X_2 < X_3)$.

(b) Now suppose that $\lambda_1 = \lambda_2 = \lambda_3 = 1/\theta$. Compare the following two estimators of θ: $\hat{\theta}_1 = (X_1 + X_2 + X_3)/3$ and $\hat{\theta}_2 = 3\min(X_1, X_2, X_3)$.

10. Let $X_1, \ldots, X_n$ be a random sample from the uniform distribution $U(\theta, \theta+1)$. To test $H_0 : \theta = 0$ versus $H_1 : \theta > 0$, we use the test

$$\text{reject } H_0, \text{ if } \min_{1\le i\le n} X_i \ge 1 \text{ or } \max_{1\le i\le n} X_i \ge c,$$

where c is a constant to be determined.

(a) Find c such that the test will have probability of type I error α.

(b) Find the power function.

(c) Is the test UMP test ? Explain.

(d) Find the values of n and c so that it will have level $\alpha = 0.1$ and power at least 0.8 if $\theta > 1$.

11. Let $X_1, \ldots, X_n$ be a random sample from the universe of Normal distribution, $N(\theta, \sigma^2)$. Answer the following questions:

   (a) Following the classical approach, we assume both θ and σ^2 are unknown constants. Derive the maximum likelihood estimators of θ and σ^2.

   (b) For simplicity, we assume σ^2 is a known constant. Following the Bayesian approach, we assume θ is a random variable with a prior distribution $N(\mu, \tau^2)$ where μ and τ^2 are known constants. Under the squared loss, derive the Bayes estimators of θ.

   (c) Compare the two estimators of θ derived above when the sample size n is small or n is large.

12. Let X_i, $i = 1, 2, \cdots, n$ be a random sample from from the pdf

$$f(x;\theta) = 3\theta^3 x^{-4}, \quad 0 < \theta < x < \infty.$$

(a) Find the maximum likelihood estimator of θ.

(b) Find the method of moments estimator of θ.

(c) Find the uniformly minimum variance unbiased estimator of θ.

Table of P(Z < z), Z ~N(0,1)

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.50000	0.50399	0.50798	0.51197	0.51595	0.51994	0.52392	0.52790	0.53188	0.53586
0.1	0.53983	0.54380	0.54776	0.55172	0.55567	0.55962	0.56356	0.56749	0.57142	0.57535
0.2	0.57926	0.58317	0.58706	0.59095	0.59483	0.59871	0.60257	0.60642	0.61026	0.61409
0.3	0.61791	0.62172	0.62552	0.62930	0.63307	0.63683	0.64058	0.64431	0.64803	0.65173
0.4	0.65542	0.65910	0.66276	0.66640	0.67003	0.67364	0.67724	0.68082	0.68439	0.68793
0.5	0.69146	0.69497	0.69847	0.70194	0.70540	0.70884	0.71226	0.71566	0.71904	0.72240
0.6	0.72575	0.72907	0.73237	0.73565	0.73891	0.74215	0.74537	0.74857	0.75175	0.75490
0.7	0.75804	0.76115	0.76424	0.76730	0.77035	0.77337	0.77637	0.77935	0.78230	0.78524
0.8	0.78814	0.79103	0.79389	0.79673	0.79955	0.80234	0.80511	0.80785	0.81057	0.81327
0.9	0.81594	0.81859	0.82121	0.82381	0.82639	0.82894	0.83147	0.83398	0.83646	0.83891
1.0	0.84134	0.84375	0.84614	0.84849	0.85083	0.85314	0.85543	0.85769	0.85993	0.86214
1.1	0.86433	0.86650	0.86864	0.87076	0.87286	0.87493	0.87698	0.87900	0.88100	0.88298
1.2	0.88493	0.88686	0.88877	0.89065	0.89251	0.89435	0.89617	0.89796	0.89973	0.90147
1.3	0.90320	0.90490	0.90658	0.90824	0.90988	0.91149	0.91309	0.91466	0.91621	0.91774
1.4	0.91924	0.92073	0.92220	0.92364	0.92507	0.92647	0.92785	0.92922	0.93056	0.93189
1.5	0.93319	0.93448	0.93574	0.93699	0.93822	0.93943	0.94062	0.94179	0.94295	0.94408
1.6	0.94520	0.94630	0.94738	0.94845	0.94950	0.95053	0.95154	0.95254	0.95352	0.95449
1.7	0.95543	0.95637	0.95728	0.95818	0.95907	0.95994	0.96080	0.96164	0.96246	0.96327
1.8	0.96407	0.96485	0.96562	0.96638	0.96712	0.96784	0.96856	0.96926	0.96995	0.97062
1.9	0.97128	0.97193	0.97257	0.97320	0.97381	0.97441	0.97500	0.97558	0.97615	0.97670
2.0	0.97725	0.97778	0.97831	0.97882	0.97932	0.97982	0.98030	0.98077	0.98124	0.98169
2.1	0.98214	0.98257	0.98300	0.98341	0.98382	0.98422	0.98461	0.98500	0.98537	0.98574
2.2	0.98610	0.98645	0.98679	0.98713	0.98745	0.98778	0.98809	0.98840	0.98870	0.98899
2.3	0.98928	0.98956	0.98983	0.99010	0.99036	0.99061	0.99086	0.99111	0.99134	0.99158
2.4	0.99180	0.99202	0.99224	0.99245	0.99266	0.99286	0.99305	0.99324	0.99343	0.99361
2.5	0.99379	0.99396	0.99413	0.99430	0.99446	0.99461	0.99477	0.99492	0.99506	0.99520
2.6	0.99534	0.99547	0.99560	0.99573	0.99585	0.99598	0.99609	0.99621	0.99632	0.99643
2.7	0.99653	0.99664	0.99674	0.99683	0.99693	0.99702	0.99711	0.99720	0.99728	0.99736
2.8	0.99744	0.99752	0.99760	0.99767	0.99774	0.99781	0.99788	0.99795	0.99801	0.99807
2.9	0.99813	0.99819	0.99825	0.99831	0.99836	0.99841	0.99846	0.99851	0.99856	0.99861
3.0	0.99865	0.99869	0.99874	0.99878	0.99882	0.99886	0.99889	0.99893	0.99896	0.99900
3.1	0.99903	0.99906	0.99910	0.99913	0.99916	0.99918	0.99921	0.99924	0.99926	0.99929
3.2	0.99931	0.99934	0.99936	0.99938	0.99940	0.99942	0.99944	0.99946	0.99948	0.99950
3.3	0.99952	0.99953	0.99955	0.99957	0.99958	0.99960	0.99961	0.99962	0.99964	0.99965
3.4	0.99966	0.99968	0.99969	0.99970	0.99971	0.99972	0.99973	0.99974	0.99975	0.99976
3.5	0.99977	0.99978	0.99978	0.99979	0.99980	0.99981	0.99981	0.99982	0.99983	0.99983
3.6	0.99984	0.99985	0.99985	0.99986	0.99986	0.99987	0.99987	0.99988	0.99988	0.99989
3.7	0.99989	0.99990	0.99990	0.99990	0.99991	0.99991	0.99992	0.99992	0.99992	0.99992
3.8	0.99993	0.99993	0.99993	0.99994	0.99994	0.99994	0.99994	0.99995	0.99995	0.99995
3.9	0.99995	0.99995	0.99996	0.99996	0.99996	0.99996	0.99996	0.99996	0.99997	0.99997
4.0	0.99997	0.99997	0.99997	0.99997	0.99997	0.99997	0.99998	0.99998	0.99998	0.99998
4.1	0.99998	0.99998	0.99998	0.99998	0.99998	0.99998	0.99998	0.99998	0.99999	0.99999