Statistics Ph.D. Qualifying Exam: Part I

October 18, 2024

Student Name: ______________________________

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1. Answer 8 out of 12 problems. Mark the problems you selected in the following table.

Problem	1	2	3	4	5	6	7	8	9	10	11	12
Selected												
Scores												

2. Write your answer right after each problem selected, attach more pages if necessary. Assemble your work in right order.

3. You can use the N(0,1) distribution table as attached.

1. Let X and Y be independent random variables with the same geometric distribution. Let U and V be defined as $U = \min(X, Y)$ and $V = X - Y$.

   (a) Show that U and V are independent.

   (b) Find the distribution of U.

2. Let X, Y, and Z be independent uniform$(0,1)$ random variables. For a constant t,

   (a) Find $P(X/Y \leq t)$.

   (b) Find $P(XY \leq t)$.

   (c) Find $P(XY/Z \leq t)$.

3. Let Y_1, Y_2, ..., Y_n be i.i.d random variable from a distribution with probability density function,

$$f(y) = \frac{\theta^2}{1+\theta}\,(1+y)\,e^{-\theta y}$$

where $y > 0$ and $\theta > 0$.

(a) Find the maximum likelihood estimator (MLE) of θ.

(b) Find the MLE of $1/\theta$.

4. Let X_1, X_2, ..., X_n be random sample from the pdf

$$f(x|\theta) = \theta x^{-2},\ 0 < \theta \leq x < \infty$$

(a) What is a sufficient statistic for θ?

(b) Find the MLE of θ.

(c) Find the method of moments estimator of θ.

5. Consider the following model: (X_i, p_i) are independent pairs with

$$\begin{aligned} &X_i|p_i \sim \text{Bernouilli}(p_i), \\ &p_i \sim \text{Beta}(\alpha, \beta), \quad i = 1, ..., n \end{aligned}$$

Let $Y = X_1 + X_2 + \ldots + X_n$.

(a) Find $\mathrm{E}(Y)$ and $\mathrm{Var}(Y)$.

(b) Find the marginal pdfs of X_i's, $i = 1, ..., n$, and use them to deduce the pdf of Y.

6. Let $X_1, ..., X_n$ be a random sample from the inverse Gaussian pdf

$$f(x|\mu, \lambda) = \sqrt{\frac{\lambda}{2\pi x^3}} \exp\left(-\frac{(\lambda - \mu)^2}{2\mu^2 x}\right), \quad x > 0$$

(a) Find the method of moments estimator of μ and λ. (Hint: $\mathrm{E}(X) = \mu$ and $\mathrm{Var}(X) = \frac{\mu^3}{\lambda}$)

(b) Find the MLE of μ and λ.

(c) Assume that λ is known. Is $\hat{\mu}$, the MLE for μ, unbiased? Does the variance of $\hat{\mu}$ attain the Rao-Cramer lower bound? Justify your answers.

7. Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from a Poisson distribution with mean λ.

   (a) Find a uniformly most powerful (UMP) test of $H_0 : \lambda \leq \lambda_0$ versus $H_1 : \lambda > \lambda_0$.

   (b) Consider the specific case $H_0 : \lambda \leq 1$ versus $H_1 : \lambda > 1$. Use the Central Limit Theorem to determine the sample size n so a UMP test satisfies $P(\text{reject } H_0 | \lambda = 1) = 0.05$ and $P(\text{reject } H_0 | \lambda = 2) = 0.9$.

8. Consider an i.i.d. sample $X_1, \ldots, X_n$ from the Pareto distribution with the pdf

$$f(x|a,b) = \frac{ba^b}{x^{b+1}}, \quad a < x < \infty, \quad a > 0, \ b > 0$$

where both parameters are unknown.

(a) Find sufficient statistics for the parameter vector (a, b).

(b) Is the Pareto family with two unknown parameters an exponential family of distributions? Justify your answer.

9. Let $X_1, X_2, \cdots, X_n$ be a random sample from the Bernouli distribution, say $P(X = 1) = \theta$ and $P(X = 0) = 1 - \theta$. We are interested in estimating $g(\theta) = \theta(1 - \theta)$.

   (a) Find the MLE of $g(\theta)$ and its asymptotic distribution.

   (b) Find the Cramer-Rao lower bound for the variance of unbiased estimators of $g(\theta)$.

   (c) Find the UMVUE of $g(\theta)$, if such exists.

10. Let $X_1, \ldots, X_n$ be a random sample from with density

$$f(x;\theta) = \frac{3x^2}{\theta^3}, \quad 0 \le x \le \theta.$$

(a) Find Method of Moments estimator of θ.

(b) Find MLE of θ.

(c) If $\theta \sim \mathcal{B}eta(2,1)$, find the posterior distribution of θ.

11. Suppose $X_1, X_2, X_3, \cdots, X_{72}$ be a random sample with a distribution whose p.d.f. is $f(x) = 2x, 0 < x < 1$.

    (a) Find the (approximate) $P(\sum_{i=1}^{72} X_i < 40)$.

    (b) Let $W = \min X_i$, a random variable representing the *minimum* value of $X_1, X_2, ..., X_{72}$. Find $P(W < 0.25)$.

12. To test $H_0 : p = 0.5$ against $H_1 : p > 0.5$, we take a random sample of Bernoulli trials $X_1, X_2, X_3, \cdots, X_n$ and use for our test statistic $Y = \sum_{i=1}^{n} X_i$. Let the critical region be defined by $C = \{y : y \geq c\}$.

   (a) If $n = 36$ and $c = 23$, find the type I error probability.

   (b) If $n = 36$ and $c = 23$, find the type II error probability when $p = 0.8$.

   (c) Find the value n and c so that the type I error probability is about 0.05 and type II error probability is about 0.2 when $p = 0.8$.

Table of P(Z < z), Z ~N(0,1)

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.50000	0.50399	0.50798	0.51197	0.51595	0.51994	0.52392	0.52790	0.53188	0.53586
0.1	0.53983	0.54380	0.54776	0.55172	0.55567	0.55962	0.56356	0.56749	0.57142	0.57535
0.2	0.57926	0.58317	0.58706	0.59095	0.59483	0.59871	0.60257	0.60642	0.61026	0.61409
0.3	0.61791	0.62172	0.62552	0.62930	0.63307	0.63683	0.64058	0.64431	0.64803	0.65173
0.4	0.65542	0.65910	0.66276	0.66640	0.67003	0.67364	0.67724	0.68082	0.68439	0.68793
0.5	0.69146	0.69497	0.69847	0.70194	0.70540	0.70884	0.71226	0.71566	0.71904	0.72240
0.6	0.72575	0.72907	0.73237	0.73565	0.73891	0.74215	0.74537	0.74857	0.75175	0.75490
0.7	0.75804	0.76115	0.76424	0.76730	0.77035	0.77337	0.77637	0.77935	0.78230	0.78524
0.8	0.78814	0.79103	0.79389	0.79673	0.79955	0.80234	0.80511	0.80785	0.81057	0.81327
0.9	0.81594	0.81859	0.82121	0.82381	0.82639	0.82894	0.83147	0.83398	0.83646	0.83891
1.0	0.84134	0.84375	0.84614	0.84849	0.85083	0.85314	0.85543	0.85769	0.85993	0.86214
1.1	0.86433	0.86650	0.86864	0.87076	0.87286	0.87493	0.87698	0.87900	0.88100	0.88298
1.2	0.88493	0.88686	0.88877	0.89065	0.89251	0.89435	0.89617	0.89796	0.89973	0.90147
1.3	0.90320	0.90490	0.90658	0.90824	0.90988	0.91149	0.91309	0.91466	0.91621	0.91774
1.4	0.91924	0.92073	0.92220	0.92364	0.92507	0.92647	0.92785	0.92922	0.93056	0.93189
1.5	0.93319	0.93448	0.93574	0.93699	0.93822	0.93943	0.94062	0.94179	0.94295	0.94408
1.6	0.94520	0.94630	0.94738	0.94845	0.94950	0.95053	0.95154	0.95254	0.95352	0.95449
1.7	0.95543	0.95637	0.95728	0.95818	0.95907	0.95994	0.96080	0.96164	0.96246	0.96327
1.8	0.96407	0.96485	0.96562	0.96638	0.96712	0.96784	0.96856	0.96926	0.96995	0.97062
1.9	0.97128	0.97193	0.97257	0.97320	0.97381	0.97441	0.97500	0.97558	0.97615	0.97670
2.0	0.97725	0.97778	0.97831	0.97882	0.97932	0.97982	0.98030	0.98077	0.98124	0.98169
2.1	0.98214	0.98257	0.98300	0.98341	0.98382	0.98422	0.98461	0.98500	0.98537	0.98574
2.2	0.98610	0.98645	0.98679	0.98713	0.98745	0.98778	0.98809	0.98840	0.98870	0.98899
2.3	0.98928	0.98956	0.98983	0.99010	0.99036	0.99061	0.99086	0.99111	0.99134	0.99158
2.4	0.99180	0.99202	0.99224	0.99245	0.99266	0.99286	0.99305	0.99324	0.99343	0.99361
2.5	0.99379	0.99396	0.99413	0.99430	0.99446	0.99461	0.99477	0.99492	0.99506	0.99520
2.6	0.99534	0.99547	0.99560	0.99573	0.99585	0.99598	0.99609	0.99621	0.99632	0.99643
2.7	0.99653	0.99664	0.99674	0.99683	0.99693	0.99702	0.99711	0.99720	0.99728	0.99736
2.8	0.99744	0.99752	0.99760	0.99767	0.99774	0.99781	0.99788	0.99795	0.99801	0.99807
2.9	0.99813	0.99819	0.99825	0.99831	0.99836	0.99841	0.99846	0.99851	0.99856	0.99861
3.0	0.99865	0.99869	0.99874	0.99878	0.99882	0.99886	0.99889	0.99893	0.99896	0.99900
3.1	0.99903	0.99906	0.99910	0.99913	0.99916	0.99918	0.99921	0.99924	0.99926	0.99929
3.2	0.99931	0.99934	0.99936	0.99938	0.99940	0.99942	0.99944	0.99946	0.99948	0.99950
3.3	0.99952	0.99953	0.99955	0.99957	0.99958	0.99960	0.99961	0.99962	0.99964	0.99965
3.4	0.99966	0.99968	0.99969	0.99970	0.99971	0.99972	0.99973	0.99974	0.99975	0.99976
3.5	0.99977	0.99978	0.99978	0.99979	0.99980	0.99981	0.99981	0.99982	0.99983	0.99983
3.6	0.99984	0.99985	0.99985	0.99986	0.99986	0.99987	0.99987	0.99988	0.99988	0.99989
3.7	0.99989	0.99990	0.99990	0.99990	0.99991	0.99991	0.99992	0.99992	0.99992	0.99992
3.8	0.99993	0.99993	0.99993	0.99994	0.99994	0.99994	0.99994	0.99995	0.99995	0.99995
3.9	0.99995	0.99995	0.99996	0.99996	0.99996	0.99996	0.99996	0.99996	0.99997	0.99997
4.0	0.99997	0.99997	0.99997	0.99997	0.99997	0.99997	0.99998	0.99998	0.99998	0.99998
4.1	0.99998	0.99998	0.99998	0.99998	0.99998	0.99998	0.99998	0.99998	0.99999	0.99999