Statistics Ph.D. Qualifying Exam: Part I

October 1, 2021

Student Name: _____

Student UID: _____

1. Answer 8 out of 12 problems. Mark the problems you selected in the following table.

Problem	1	2	3	4	5	6	7	8	9	10	11	12
Selected												
Scores												

2. Write your answer right after each problem selected, attach more pages if necessary. Assemble your work in right order.

3. You can use the N(0,1) distribution table as attached.

1. Let X_1 and X_2 be independent random variables. Assume that X_1 follows a binomial distribution $B(50, 0.4)$ and X_2 follows a Poisson distribution with mean 30, respectively. Let $Y = X_1 + X_2$ be the sum of two random variables.

   (a) Find the moment generating function of Y.

   (b) Find $E(Y)$ and $Var(Y)$.

   (c) Find/compare the exact and approximation of $P(X_1 = 20)$.

   (d) Approximate $P(45 \leq Y \leq 55)$.

   (e) Approximate $P(X_1 \leq X_2)$.

2. Let X_1 and X_2 be two independent random variables, each with a with chi-squared distributions of degrees of freedom, r_1 and r_2, respectively.

$$Y_1 = \frac{X_1}{X_1 + X_2}, \quad Y_2 = X_1 + X_2.$$

(a) Find the joint pdf of Y_1 and Y_2, $g(x, y)$.

(b) Find the marginal p.d.f of Y_1, $g_1(y)$. Name the distribution of Y_1.

(c) Find the marginal p.d.f of Y_2, $g_2(y)$. Name the distribution of Y_2.

(d) Name the distribution of $W = \frac{X_1}{X_2}$. (no derivation is necessary).

3. Let $X_1, \ldots, X_n$ be a random sample of size n from a population with density

$$f(x;\theta) = \frac{1}{\theta} e^{-x/\theta}, \quad x > 0, \theta > 0.$$

We wish to estimate $\tau = P(X_1 > 1) = e^{-1/\theta}$.

(a) Compute the Cramer-Rao lower bound for the variance of unbiased estimator of τ.

(b) Find the maximum likelihood estimator of τ.

(c) Find the uniformly minimum variance unbiased (UMVU) estimator of τ.

4. If $X_1, X_2, \ldots, X_n$ is a random sample from a standard normal distribution ($N(0,1)$). Let

$$\bar{X} = \frac{\sum_{i=1}^{n} X_i}{n} \qquad S^2 = \frac{\sum_{i=1}^{n} (X_i - \bar{X})^2}{n-1}$$

   (a) Find the distribution of $\bar{X}$.

   (b) Find the distribution of S^2.

   (c) Show that $\bar{X}$ and S^2 are independent.

5. Let U_1, U_2 be a random sample of size 2 from U(0,1), a uniform distribution over the interval (0,1). Let $Y_{(1)}$ and $Y_{(2)}$ be the corresponding order statistics.

   (a) Find the conditional distribution of $Y_{(1)}$ given $Y_{(2)} = y_2$, $f_{Y_{(1)}|Y_{(2)}}(y_1|y_2)$.

   (b) What is the distribution of $Y_{(2)} - Y_{(1)}$?

6. Let $X_1, X_2, \ldots, X_n$ be a random sample from $f(x;\theta) = \theta e^{-\theta x}$. Find a most powerful test of $H_0 : \theta = \theta_0$ versus $H_1 : \theta = \theta_1$ where $\theta_1 > \theta_0$. Explain how you could find the rejection region for the test.

7. Let (X_1, Y_1), ..., (X_n, Y_n) be a random sample from a bivariate distribution. The conditional distribution of X_i given $Y_i = y_i$ is $N(y_i, 1)$ and the distribution for Y_i is $N(0, e^{\theta})$.

   (a) What is the distribution of $X_i - Y_i$?

   (b) Are $X_i - Y_i$ and Y_i independent? Justify your answer.

   (c) What is the distribution of $\sum_{i=1}^{n}(X_i - Y_i)^2 + \sum_{i=1}^{n}(e^{-\theta/2}Y_i)^2$?

8. Let X_1, ..., X_n be a random sample drawn from a Poisson distribution with mean λ,

$$f(x|\lambda) = \frac{\lambda^x e^{-\lambda}}{x!}$$

where λ is an unknown parameter. Consider two estimators for λ, $T_1 = \sum_{i=1}^{n} X_i$ and $T_2 = \sum_{i=1}^{n} (X_i - \bar{X})^2$

(a) Are T_1 and T_2 unbiased estimators for λ? Justify your answer.

(b) Which estimator is more efficient for λ? Justify your answer.

9. Let (x_1, Y_1), ..., (x_n, Y_n) be n pairs of independent samples. Suppose $Y_i \sim Exponential(\mu = \theta x_i)$, that is,

$$f_{Y_i}(y_i|\theta, x_i) = \frac{1}{\theta x_i} e^{-y_i/\theta x_i}$$

where $\theta > 0$ is an unkown parameter, $i = 1, ..., n$ and $x_1, ..., x_n$ are positive known constants. Define $T_n = \sum_{i=1}^{n} \frac{Y_i}{x_i}$.

(a) Write the likelihood for θ as a function of $T_n, x_1, ..., x_n$ and θ.

(b) Is T_n complete and sufficient for θ? why or why not?

(c) Find the uniform minimum variance unbiased estimator (UMVUE) for θ.

10. Let X, Y, U be independent random variables with $X \sim \text{Poisson}(\lambda)$, $Y \sim \text{Poisson}(\mu)$, and $U \sim \text{Uniform}(0, 1)$ Let

$$V = \begin{cases} X & , \text{ if } U > a \\ Y & , \text{ if } U \leq a, \end{cases}$$

Find

$$P(X + Y = n \mid V = k)$$

11. Let $X_1, \ldots, X_m$ and $Y_1, \ldots, Y_n$ be independent samples from Geometric $(\alpha\beta)$ and Poisson(β) populations respectively.

    (a) Find the MLE's $(\hat{\alpha}, \hat{\beta})$ for (α, β).

    (b) Find jointly sufficient statistics S for (α, β).

    (c) Is $(\hat{\alpha}, \hat{\beta})$ a function of S?

    (d) Is $(\hat{\alpha}, \hat{\beta}$ jointly minimally sufficient for (α, β)?

12. It is hypothesized that in Paris, Tennessee, 10% of all households have 3PC's, 20% have 2PC's, 40% have 1PC, and 30% have no PC's. In a survey of the households in Paris, a sample of 100 households is randomly selected. Assuming that the hypothesis is true, what is the probability that the **average number** of PC's among the selected households is less than 1.5?

Table of P(Z < z), Z ~N(0,1)

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.50000	0.50399	0.50798	0.51197	0.51595	0.51994	0.52392	0.52790	0.53188	0.53586
0.1	0.53983	0.54380	0.54776	0.55172	0.55567	0.55962	0.56356	0.56749	0.57142	0.57535
0.2	0.57926	0.58317	0.58706	0.59095	0.59483	0.59871	0.60257	0.60642	0.61026	0.61409
0.3	0.61791	0.62172	0.62552	0.62930	0.63307	0.63683	0.64058	0.64431	0.64803	0.65173
0.4	0.65542	0.65910	0.66276	0.66640	0.67003	0.67364	0.67724	0.68082	0.68439	0.68793
0.5	0.69146	0.69497	0.69847	0.70194	0.70540	0.70884	0.71226	0.71566	0.71904	0.72240
0.6	0.72575	0.72907	0.73237	0.73565	0.73891	0.74215	0.74537	0.74857	0.75175	0.75490
0.7	0.75804	0.76115	0.76424	0.76730	0.77035	0.77337	0.77637	0.77935	0.78230	0.78524
0.8	0.78814	0.79103	0.79389	0.79673	0.79955	0.80234	0.80511	0.80785	0.81057	0.81327
0.9	0.81594	0.81859	0.82121	0.82381	0.82639	0.82894	0.83147	0.83398	0.83646	0.83891
1.0	0.84134	0.84375	0.84614	0.84849	0.85083	0.85314	0.85543	0.85769	0.85993	0.86214
1.1	0.86433	0.86650	0.86864	0.87076	0.87286	0.87493	0.87698	0.87900	0.88100	0.88298
1.2	0.88493	0.88686	0.88877	0.89065	0.89251	0.89435	0.89617	0.89796	0.89973	0.90147
1.3	0.90320	0.90490	0.90658	0.90824	0.90988	0.91149	0.91309	0.91466	0.91621	0.91774
1.4	0.91924	0.92073	0.92220	0.92364	0.92507	0.92647	0.92785	0.92922	0.93056	0.93189
1.5	0.93319	0.93448	0.93574	0.93699	0.93822	0.93943	0.94062	0.94179	0.94295	0.94408
1.6	0.94520	0.94630	0.94738	0.94845	0.94950	0.95053	0.95154	0.95254	0.95352	0.95449
1.7	0.95543	0.95637	0.95728	0.95818	0.95907	0.95994	0.96080	0.96164	0.96246	0.96327
1.8	0.96407	0.96485	0.96562	0.96638	0.96712	0.96784	0.96856	0.96926	0.96995	0.97062
1.9	0.97128	0.97193	0.97257	0.97320	0.97381	0.97441	0.97500	0.97558	0.97615	0.97670
2.0	0.97725	0.97778	0.97831	0.97882	0.97932	0.97982	0.98030	0.98077	0.98124	0.98169
2.1	0.98214	0.98257	0.98300	0.98341	0.98382	0.98422	0.98461	0.98500	0.98537	0.98574
2.2	0.98610	0.98645	0.98679	0.98713	0.98745	0.98778	0.98809	0.98840	0.98870	0.98899
2.3	0.98928	0.98956	0.98983	0.99010	0.99036	0.99061	0.99086	0.99111	0.99134	0.99158
2.4	0.99180	0.99202	0.99224	0.99245	0.99266	0.99286	0.99305	0.99324	0.99343	0.99361
2.5	0.99379	0.99396	0.99413	0.99430	0.99446	0.99461	0.99477	0.99492	0.99506	0.99520
2.6	0.99534	0.99547	0.99560	0.99573	0.99585	0.99598	0.99609	0.99621	0.99632	0.99643
2.7	0.99653	0.99664	0.99674	0.99683	0.99693	0.99702	0.99711	0.99720	0.99728	0.99736
2.8	0.99744	0.99752	0.99760	0.99767	0.99774	0.99781	0.99788	0.99795	0.99801	0.99807
2.9	0.99813	0.99819	0.99825	0.99831	0.99836	0.99841	0.99846	0.99851	0.99856	0.99861
3.0	0.99865	0.99869	0.99874	0.99878	0.99882	0.99886	0.99889	0.99893	0.99896	0.99900
3.1	0.99903	0.99906	0.99910	0.99913	0.99916	0.99918	0.99921	0.99924	0.99926	0.99929
3.2	0.99931	0.99934	0.99936	0.99938	0.99940	0.99942	0.99944	0.99946	0.99948	0.99950
3.3	0.99952	0.99953	0.99955	0.99957	0.99958	0.99960	0.99961	0.99962	0.99964	0.99965
3.4	0.99966	0.99968	0.99969	0.99970	0.99971	0.99972	0.99973	0.99974	0.99975	0.99976
3.5	0.99977	0.99978	0.99978	0.99979	0.99980	0.99981	0.99981	0.99982	0.99983	0.99983
3.6	0.99984	0.99985	0.99985	0.99986	0.99986	0.99987	0.99987	0.99988	0.99988	0.99989
3.7	0.99989	0.99990	0.99990	0.99990	0.99991	0.99991	0.99992	0.99992	0.99992	0.99992
3.8	0.99993	0.99993	0.99993	0.99994	0.99994	0.99994	0.99994	0.99995	0.99995	0.99995
3.9	0.99995	0.99995	0.99996	0.99996	0.99996	0.99996	0.99996	0.99996	0.99997	0.99997
4.0	0.99997	0.99997	0.99997	0.99997	0.99997	0.99997	0.99998	0.99998	0.99998	0.99998
4.1	0.99998	0.99998	0.99998	0.99998	0.99998	0.99998	0.99998	0.99998	0.99999	0.99999