Statistics Ph.D. Qualifying Exam: Part I

August 5, 2016

Student Name: _____

1. Answer 8 out of 12 problems. Mark the problems you selected in the following table.

Problem	1	2	3	4	5	6	7	8	9	10	11	12
Selected												
Scores												

2. Write your answer right after each problem selected, attach more pages if necessary. **Do not** write your answers on the back.

3. Assemble your work in right order and in the original problem order. (Including the ones that you do not select)

4. You can use the N(0,1) distribution table as attached.

1. Let U_1, U_2, U_3 be iid Bernoulli with parameter π, that is

$$P(U_i = u_i) = \pi^{u_i}(1-\pi)^{1-u_i} \quad u_i = 0, 1 \;\; i = 1, 2, 3 \;\; 0 < \pi < 1$$

Let W be independent of U_i, $i = 1, 2, 3$ with a Bernoulli distribution with parameter θ, that is

$$P(W = w) = \theta^w(1-\theta)^{1-w} \;\; w = 0, 1 \quad 0 < \theta < 1.$$

Let $X = WU_1 + (1-W)U_2$ and $Y = WU_1 + (1-W)U_3$.

(a) Find the correlation between X and Y.

(b) Find $E(Y|X = x)$.

2. $X_1, X_2, \ldots, X_n$ are iid with pdf given by

$$P(X = x; \theta) = (1 - \theta)\theta^{x-1} \quad x = 1, 2, \ldots \quad 0 < \theta < 1.$$

(a) Find the UMVUE of $1/\theta$.

(b) Find the uniformly most powerful (UMP) test of size α for testing $H_0 : \theta = 1/3$ versus the alternative hypothesis $H_1 : \theta > 1/3$.

3. $X_1, X_2, \ldots, X_n$ is a random sample from density

$$f_X(x;\theta) = \theta e^{-\theta x} \quad x > 0 \quad \theta > 0.$$

(a) Find the UMVUE of θ.

(b) Find the Cramer Rao lower bound of any unbiased estimator of θ.

(c) Find the mean squared error of the UMVUE.

4. X and Y have joint density

$$f_{XY}(x,y) = e^{-(\theta x + \theta^{-1} y)} \quad x > 0, y > 0, \theta > 0$$

(a) Find the joint pdf of $U = \left(\frac{Y}{X}\right)^{1/2}$ and $V = (XY)^{1/2}$.

(b) Find the marginal distribution of U.

(c) Find $E(U)$.

5. Let $X_1, X_2, \cdots X_n$ be independent N(0,1) and let $u_4 = E(X_1^4)$.

   (a) Find u_4.

   (b) Show that, as $n \to \infty$

$$\frac{\sum_{i=1}^{n}(X_i^2 - 1)}{\sqrt{n(u_4 - 1)}} \to N(0, 1).$$

6. Consider independent random samples from two exponential distributions, $X_i \sim EXP(\theta_1)$ and $Y_j \sim EXP(\theta_2)$; $i = 1, \cdots, n_1$, $j = 1, \cdots, n_2$.

   (a) Show that $(\theta_2/\theta_1)(\bar{X}/\bar{Y}) \sim F(2n_1, 2n_2)$.

   (b) Derive a $100\gamma\%$ confidence interval for θ_2/θ_1.

7. Let

$$Y_i = \beta x_i + \epsilon_i,$$

where x_i is a fixed covariate and $\epsilon_i \sim N(0, \sigma_0^2), i = 1, \ldots, n$ where σ_0^2 is known and β is an unknown parameter.

(a) Find the maximum likelihood estimator of β.

(b) Using a $N(0, 1)$ as a prior distribution for β, find the posterior distribution of β.

(c) How does this posterior distribution relate to the MLE for β?

8. Let $X_1, X_2 \ldots$ be a sequence of independent identically distributed exponential random variables with mean $\frac{1}{\theta}$. Let $a > 0$ be a fixed number and let $N = \min\{n : X_n > a\}$

   (a) By explicit derivation, find and name the distribution of N.

   (b) Find $E(X_N)$.

9. Let $X_1, \ldots, X_n$ be a random sample from a population with unknown parameter θ.

   (a) If the population has a Normal$(0, \theta)$ distribution, find the maximum likelihood estimator of θ.

   (b) Is the MLE of θ a sufficient statistic for θ?

   (c) If the population has a Uniform $(0, \theta)$ distribution find the maximum likelihood estimator of θ.

   (d) Is the MLE of θ a sufficient statistic for θ?

10. An urn contains seven balls, THREE marked WIN and FOUR marked LOSE. You and another player take turns selecting a ball from the urn, one at a time. The first person to select the LAST (third) WIN ball is the winner. If you draw fist, find the probability that you will win if the sampling is done

    (a) Without replacement.

    (b) With replacement.

11. Let $X_1, \ldots, X_n$ be i.i.d. from the uniform distribution on the interval $(\theta, \theta + 1)$.

    (a) Find the joint distribution of $X_{(1)}$ and $X_{(n)}$, where $X_{(1)} = \min(X_1, \ldots, X_n)$, and $X_{(n)} = \max(X_1, \ldots, X_n)$.

    (b) Find MLE of θ.

    (c) Find a UMP test of size α for testing $H_0 : \theta \leq 0$ versus $H_1 : \theta > 0$.

12. Let X and Y have a trinomial distribution with parameters $n = 4$, $p_1 = 1/4$ and $p_2 = 1/2$. Using the distributional property of trinomial distribution, answer the following:

    (a) Write the joint p.m.f. of X and Y, $f(x, y) = P(X = x, Y = y)$.

    (b) What is the marginal distribution of X ?

    (c) What is the marginal distribution of Y ?

    (d) Find $Var(Y|x)$, the conditional variance of Y, given that $X = x$.

Table of P(Z < z), Z ~N(0,1)

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.50000	0.50399	0.50798	0.51197	0.51595	0.51994	0.52392	0.52790	0.53188	0.53586
0.1	0.53983	0.54380	0.54776	0.55172	0.55567	0.55962	0.56356	0.56749	0.57142	0.57535
0.2	0.57926	0.58317	0.58706	0.59095	0.59483	0.59871	0.60257	0.60642	0.61026	0.61409
0.3	0.61791	0.62172	0.62552	0.62930	0.63307	0.63683	0.64058	0.64431	0.64803	0.65173
0.4	0.65542	0.65910	0.66276	0.66640	0.67003	0.67364	0.67724	0.68082	0.68439	0.68793
0.5	0.69146	0.69497	0.69847	0.70194	0.70540	0.70884	0.71226	0.71566	0.71904	0.72240
0.6	0.72575	0.72907	0.73237	0.73565	0.73891	0.74215	0.74537	0.74857	0.75175	0.75490
0.7	0.75804	0.76115	0.76424	0.76730	0.77035	0.77337	0.77637	0.77935	0.78230	0.78524
0.8	0.78814	0.79103	0.79389	0.79673	0.79955	0.80234	0.80511	0.80785	0.81057	0.81327
0.9	0.81594	0.81859	0.82121	0.82381	0.82639	0.82894	0.83147	0.83398	0.83646	0.83891
1.0	0.84134	0.84375	0.84614	0.84849	0.85083	0.85314	0.85543	0.85769	0.85993	0.86214
1.1	0.86433	0.86650	0.86864	0.87076	0.87286	0.87493	0.87698	0.87900	0.88100	0.88298
1.2	0.88493	0.88686	0.88877	0.89065	0.89251	0.89435	0.89617	0.89796	0.89973	0.90147
1.3	0.90320	0.90490	0.90658	0.90824	0.90988	0.91149	0.91309	0.91466	0.91621	0.91774
1.4	0.91924	0.92073	0.92220	0.92364	0.92507	0.92647	0.92785	0.92922	0.93056	0.93189
1.5	0.93319	0.93448	0.93574	0.93699	0.93822	0.93943	0.94062	0.94179	0.94295	0.94408
1.6	0.94520	0.94630	0.94738	0.94845	0.94950	0.95053	0.95154	0.95254	0.95352	0.95449
1.7	0.95543	0.95637	0.95728	0.95818	0.95907	0.95994	0.96080	0.96164	0.96246	0.96327
1.8	0.96407	0.96485	0.96562	0.96638	0.96712	0.96784	0.96856	0.96926	0.96995	0.97062
1.9	0.97128	0.97193	0.97257	0.97320	0.97381	0.97441	0.97500	0.97558	0.97615	0.97670
2.0	0.97725	0.97778	0.97831	0.97882	0.97932	0.97982	0.98030	0.98077	0.98124	0.98169
2.1	0.98214	0.98257	0.98300	0.98341	0.98382	0.98422	0.98461	0.98500	0.98537	0.98574
2.2	0.98610	0.98645	0.98679	0.98713	0.98745	0.98778	0.98809	0.98840	0.98870	0.98899
2.3	0.98928	0.98956	0.98983	0.99010	0.99036	0.99061	0.99086	0.99111	0.99134	0.99158
2.4	0.99180	0.99202	0.99224	0.99245	0.99266	0.99286	0.99305	0.99324	0.99343	0.99361
2.5	0.99379	0.99396	0.99413	0.99430	0.99446	0.99461	0.99477	0.99492	0.99506	0.99520
2.6	0.99534	0.99547	0.99560	0.99573	0.99585	0.99598	0.99609	0.99621	0.99632	0.99643
2.7	0.99653	0.99664	0.99674	0.99683	0.99693	0.99702	0.99711	0.99720	0.99728	0.99736
2.8	0.99744	0.99752	0.99760	0.99767	0.99774	0.99781	0.99788	0.99795	0.99801	0.99807
2.9	0.99813	0.99819	0.99825	0.99831	0.99836	0.99841	0.99846	0.99851	0.99856	0.99861
3.0	0.99865	0.99869	0.99874	0.99878	0.99882	0.99886	0.99889	0.99893	0.99896	0.99900
3.1	0.99903	0.99906	0.99910	0.99913	0.99916	0.99918	0.99921	0.99924	0.99926	0.99929
3.2	0.99931	0.99934	0.99936	0.99938	0.99940	0.99942	0.99944	0.99946	0.99948	0.99950
3.3	0.99952	0.99953	0.99955	0.99957	0.99958	0.99960	0.99961	0.99962	0.99964	0.99965
3.4	0.99966	0.99968	0.99969	0.99970	0.99971	0.99972	0.99973	0.99974	0.99975	0.99976
3.5	0.99977	0.99978	0.99978	0.99979	0.99980	0.99981	0.99981	0.99982	0.99983	0.99983
3.6	0.99984	0.99985	0.99985	0.99986	0.99986	0.99987	0.99987	0.99988	0.99988	0.99989
3.7	0.99989	0.99990	0.99990	0.99990	0.99991	0.99991	0.99992	0.99992	0.99992	0.99992
3.8	0.99993	0.99993	0.99993	0.99994	0.99994	0.99994	0.99994	0.99995	0.99995	0.99995
3.9	0.99995	0.99995	0.99996	0.99996	0.99996	0.99996	0.99996	0.99996	0.99997	0.99997
4.0	0.99997	0.99997	0.99997	0.99997	0.99997	0.99997	0.99998	0.99998	0.99998	0.99998
4.1	0.99998	0.99998	0.99998	0.99998	0.99998	0.99998	0.99998	0.99998	0.99999	0.99999