Statistics Ph.D. Qualifying Exam: Part II

August 16, 2013

Student Name: ______________________________

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1. Answer 8 out of 12 problems. Mark the problems you selected in the following table.

Problem	1	2	3	4	5	6	7	8	9	10	11	12
Selected												
Scores												

2. Write your answer right after each problem selected, attach more pages if necessary. **Do not** write your answers on the back.

3. Assemble your work in right order and in the original problem order. (Including the ones that you do not select)

4. You can use the N(0,1) distribution table as attached.

1. Let $X_1, X_2 \ldots$ be independent and identically distributed continuous random variables. A record occurs at time $n, n > 0$ and has value X_n if $X_n > \max(X_1, \ldots X_{n-1})$, where $X_0 = -\infty$.

   (a) Let N_n denote the total number of records that have occurred up to and including time n. Compute $E(N_n)$ and $Var(N_n)$.

   (b) Let $T = \min(n : n > 1$ and a record occurs at $n)$. Compute $P(T > n)$ and use it to show that $E(T) = \infty$.

2. Let $X_1, \ldots, X_n$ be a random sample of size n from $f(x;\theta) = \theta^2 x e^{-\theta x} \quad x > 0$.

   (a) In testing $H_0 : \theta \leq 1$ versus $H_1 : \theta > 1$ the following test was used: Reject H_0 if $X_1 \leq 1$. Find the power function and size of this test.

   (b) Find a most powerful size α test of $H_0 : \theta = 1$ versus $H_1 : \theta = 2$.

   (c) Does there exist a uniformly most powerful size α test of $H_0 : \theta \leq 1$ versus $H_1 : \theta > 1$? If so, what is the rejection region?

   (d) In testing $H_0 : \theta = 1$ versus $H_1 : \theta = 2$, among all tests based on the likelihood ratio of the form $\dfrac{L(\theta_{H_0})}{L(\theta_{H_1})}$ find a test that minimizes the sum of the sizes of the Type 1 and Type II errors based on a sample of size $n = 1$.

3. Let X_1 and X_2 be independent standard normal random variables

   (a) Derive the joint distribution of $Y_1 = X_1^2 + X_2^2$ and $Y_2 = X_2$

   (b) Derive the marginal distribution of Y_1.

4. Let $X_1, \ldots X_n$ be a random sample from each of the distributions below. In each case, find the UMVUE of θ^r.

   (a) $f(x;\theta) = \dfrac{1}{\theta} \quad 0 < x < \theta,\ r < n.$

   (b) $f(x;\theta) = e^{-(x-\theta)}, x > \theta$

5. Let $\{X_1, \ldots, X_n\}$ be a random sample from a population with density $f(x, \theta) = \{\theta^2 + 2\theta\,(1-\theta)\}^x (1-\theta)^{2(1-x)}, x = 0, 1, 0 < \theta < 1$.

   (a) Find the moment estimator of θ.

   (b) Derive the MLE of θ.

   (c) Explain how to use the EM-algorithm to find the MLE of θ.

   (d) Compare above methods to estimate θ.

6. Let $X_1, \ldots, X_n$ be a sample from a population with p.d.f. $f(x;\theta) = e^{-\theta}\theta^x/x!$ where $x = 0, 1, 2, \ldots$. We would like to test $H_0 : \theta = 20$ and $H_1 : \theta = 10$.

   (a) Derive the most powerful (MP) test.

   (b) With a small n (say, $n = 5$), is it adequate to use the normal approximation for the distribution of $\sum_{i=1}^n X_i$ under either $H_0 : \theta = 20$ or $H_1 : \theta = 10$? Justify your answer.

   (c) Explain how to determine the sample size n to guarantee both type I and type II errors are about 0.05.

7. Let $X_1, X_2, \ldots, X_n$ be a random sample from $U(\theta_1, \theta_2)$, where both θ_1, θ_2 are unknown. Find the UMVUE (uniformly minimum variance unbiased estimator) for

   (a) $\theta_1 + \theta_2$.

   (b) $\theta_2 - \theta_1$.

   (c) $(\theta_2 - \theta_1)^c$, where $c > 0$ is a known constant.

8. Let $X_1, X_2, \ldots, X_n$ be a random sample from $f(x; \theta) = \theta(1-\theta)^x, x = 0, 1, 2, ..., 0 < \theta < 1$.

   (a) Derive $E(X)$.

   (b) Obtain the MLE of $E(X)$.

   (c) Find the UMVUE (uniformly minimum variance unbiased estimator) of $E(X)$ if such exists.

   (d) Obtain the Cramer-Rao lower bound for the unbiased estimator of $E(X)$.

9. Suppose that $Y_{ij} \sim N(\mu_i, \sigma^2)$ for $i = 1, \ldots, m$ and $j = 1, \ldots, n$, where $\mu_i, \sigma^2, i = 1, \ldots, m$ are unknown parameters.

   (a) Find the MLE of $\mu_i, i = 1, \ldots, n$ and σ^2.

   (b) Representing $Y_{ij}, i = 1, \ldots, m$ and $j = 1, \ldots, n$, as vector Y, and $\mu_i, i = 1, \ldots, n$ as vector μ, express the relationship between Y and μ as a linear model

$$Y = \; X\mu + \epsilon$$

   stating what the matrix X should be.

   (c) Suppose we put the following prior distribution on $(\mu, \frac{1}{\sigma^2})$:

$$\begin{aligned} \mu|\sigma^2 &\sim N(\mu_0.\frac{V_0}{\sigma^2}) \\ \frac{1}{\sigma^2} &\sim Gamma(\alpha_0, \beta_0). \end{aligned}$$

   Find the posterior distributions of $\mu|\sigma^2$ and $\frac{1}{\sigma^2}$.

10. Let $\{(Y_i, x_i); i = 1, \ldots, n\}$ satisfy the regression model

$$Y_i = \beta x_i + \epsilon_i,$$

where $E(\epsilon_i) = 0$, $\mathrm{Var}(\epsilon_i) = \sigma^2$ and ϵ_i's are independent.

(a) Find $\hat{\beta}$ the least squares estimator of β.

(b) Let $\hat{\sigma}^2 = \frac{1}{n-1}\sum_{i=1}^{n}(Y_i - \hat{\beta})^2$. Is $\hat{\sigma}^2$ an unbiased estimator of σ^2? (Give a proof of your answer.)

(c) Are $\hat{\beta}$ and $\hat{\sigma}^2$ independent? (Give a proof of your answer.)

(d) Construct a test with level of significance α for testing $H_0 : \beta = 0$ versus $H_1 : \beta \neq 0$. and state the properties of your test.

(e) Prove that, with an appropriate choice of scaling factor a_n,

$$a_n \frac{\hat{\beta} - \beta}{\hat{\sigma}}$$

converge in distribution to a $N(0, 1)$ distribution.

11. Let $X_1, \ldots, X_n$ be a random sample from Uniform $(0, \theta)$, where $\theta > 0$. Suppose that we put the Gamma prior density

$$\pi(\theta|\gamma) = \frac{\theta^n}{\gamma^{n+1}\Gamma(n+1)} e^{-\frac{\theta}{\gamma}}, \quad \theta > 0.$$

on θ. Let $Y = X_{(n)} = \max\{X_1, \ldots, X_n\}$

(a) Find the density $g(y|\theta)$ of Y given θ.

(b) Find the marginal (unconditional) density of Y.

(c) Find the density $q(\theta|Y = y)$ of θ given Y.

(d) Find the Bayes estimator of θ using squared error loss function.

(e) Find the maximum likelihood estimator of θ.

(f) Compare the mean squared errors of the Bayes estimator and the MLE.

12. Let $X = (X_1, \ldots, X_n)$ be a multinomial vector with parameters n, θ, where $\theta = (\theta_1, \ldots, \theta_K), K < n$. Suppose θ has Dirichlet prior density

$$\pi(\theta) = \frac{\Gamma(\alpha_1 + \cdots + \alpha_K)}{\prod_{j=1}^{K} \Gamma \alpha_j} \theta_1^{\alpha_1 - 1} \cdots \theta_K^{\alpha_K - 1},$$

where $\sum_{i=1}^{K} \theta_i = 1$.

(a) Find the posterior distribution of θ.

(b) Using the loss function $L(\theta,\ d) = \sum_{i=1}^{K} (\theta_i - d_i)^2$, show that the Bayes estimator of θ is given by

$$d_0(\ X) = E(\theta|\ X) = \frac{\alpha +\ X}{\sum_{i=1}^{K} \alpha_i + n}.$$

(c) Find the mode of the posterior distribution of θ and compare it with the MLE of θ.

(d) Find the limiting distribution of $\sqrt{n}(\ d_0(\ X) - \theta)$ as $n \to \infty$.

Table of P(Z < z), Z ~N(0,1)

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.50000	0.50399	0.50798	0.51197	0.51595	0.51994	0.52392	0.52790	0.53188	0.53586
0.1	0.53983	0.54380	0.54776	0.55172	0.55567	0.55962	0.56356	0.56749	0.57142	0.57535
0.2	0.57926	0.58317	0.58706	0.59095	0.59483	0.59871	0.60257	0.60642	0.61026	0.61409
0.3	0.61791	0.62172	0.62552	0.62930	0.63307	0.63683	0.64058	0.64431	0.64803	0.65173
0.4	0.65542	0.65910	0.66276	0.66640	0.67003	0.67364	0.67724	0.68082	0.68439	0.68793
0.5	0.69146	0.69497	0.69847	0.70194	0.70540	0.70884	0.71226	0.71566	0.71904	0.72240
0.6	0.72575	0.72907	0.73237	0.73565	0.73891	0.74215	0.74537	0.74857	0.75175	0.75490
0.7	0.75804	0.76115	0.76424	0.76730	0.77035	0.77337	0.77637	0.77935	0.78230	0.78524
0.8	0.78814	0.79103	0.79389	0.79673	0.79955	0.80234	0.80511	0.80785	0.81057	0.81327
0.9	0.81594	0.81859	0.82121	0.82381	0.82639	0.82894	0.83147	0.83398	0.83646	0.83891
1.0	0.84134	0.84375	0.84614	0.84849	0.85083	0.85314	0.85543	0.85769	0.85993	0.86214
1.1	0.86433	0.86650	0.86864	0.87076	0.87286	0.87493	0.87698	0.87900	0.88100	0.88298
1.2	0.88493	0.88686	0.88877	0.89065	0.89251	0.89435	0.89617	0.89796	0.89973	0.90147
1.3	0.90320	0.90490	0.90658	0.90824	0.90988	0.91149	0.91309	0.91466	0.91621	0.91774
1.4	0.91924	0.92073	0.92220	0.92364	0.92507	0.92647	0.92785	0.92922	0.93056	0.93189
1.5	0.93319	0.93448	0.93574	0.93699	0.93822	0.93943	0.94062	0.94179	0.94295	0.94408
1.6	0.94520	0.94630	0.94738	0.94845	0.94950	0.95053	0.95154	0.95254	0.95352	0.95449
1.7	0.95543	0.95637	0.95728	0.95818	0.95907	0.95994	0.96080	0.96164	0.96246	0.96327
1.8	0.96407	0.96485	0.96562	0.96638	0.96712	0.96784	0.96856	0.96926	0.96995	0.97062
1.9	0.97128	0.97193	0.97257	0.97320	0.97381	0.97441	0.97500	0.97558	0.97615	0.97670
2.0	0.97725	0.97778	0.97831	0.97882	0.97932	0.97982	0.98030	0.98077	0.98124	0.98169
2.1	0.98214	0.98257	0.98300	0.98341	0.98382	0.98422	0.98461	0.98500	0.98537	0.98574
2.2	0.98610	0.98645	0.98679	0.98713	0.98745	0.98778	0.98809	0.98840	0.98870	0.98899
2.3	0.98928	0.98956	0.98983	0.99010	0.99036	0.99061	0.99086	0.99111	0.99134	0.99158
2.4	0.99180	0.99202	0.99224	0.99245	0.99266	0.99286	0.99305	0.99324	0.99343	0.99361
2.5	0.99379	0.99396	0.99413	0.99430	0.99446	0.99461	0.99477	0.99492	0.99506	0.99520
2.6	0.99534	0.99547	0.99560	0.99573	0.99585	0.99598	0.99609	0.99621	0.99632	0.99643
2.7	0.99653	0.99664	0.99674	0.99683	0.99693	0.99702	0.99711	0.99720	0.99728	0.99736
2.8	0.99744	0.99752	0.99760	0.99767	0.99774	0.99781	0.99788	0.99795	0.99801	0.99807
2.9	0.99813	0.99819	0.99825	0.99831	0.99836	0.99841	0.99846	0.99851	0.99856	0.99861
3.0	0.99865	0.99869	0.99874	0.99878	0.99882	0.99886	0.99889	0.99893	0.99896	0.99900
3.1	0.99903	0.99906	0.99910	0.99913	0.99916	0.99918	0.99921	0.99924	0.99926	0.99929
3.2	0.99931	0.99934	0.99936	0.99938	0.99940	0.99942	0.99944	0.99946	0.99948	0.99950
3.3	0.99952	0.99953	0.99955	0.99957	0.99958	0.99960	0.99961	0.99962	0.99964	0.99965
3.4	0.99966	0.99968	0.99969	0.99970	0.99971	0.99972	0.99973	0.99974	0.99975	0.99976
3.5	0.99977	0.99978	0.99978	0.99979	0.99980	0.99981	0.99981	0.99982	0.99983	0.99983
3.6	0.99984	0.99985	0.99985	0.99986	0.99986	0.99987	0.99987	0.99988	0.99988	0.99989
3.7	0.99989	0.99990	0.99990	0.99990	0.99991	0.99991	0.99992	0.99992	0.99992	0.99992
3.8	0.99993	0.99993	0.99993	0.99994	0.99994	0.99994	0.99994	0.99995	0.99995	0.99995
3.9	0.99995	0.99995	0.99996	0.99996	0.99996	0.99996	0.99996	0.99996	0.99997	0.99997
4.0	0.99997	0.99997	0.99997	0.99997	0.99997	0.99997	0.99998	0.99998	0.99998	0.99998
4.1	0.99998	0.99998	0.99998	0.99998	0.99998	0.99998	0.99998	0.99998	0.99999	0.99999