# Statistics Ph.D. Qualifying Exam: Part II

November 12, 2011

Student Name: ______________________________

1. Answer 8 out of 12 problems. Mark the problems you selected in the following table.

| Problem | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Selected | | | | | | | | | | | | |
| Scores | | | | | | | | | | | | |

2. Write your answer right after each problem selected, attach more pages if necessary. **Do not** write your answers on the back.

3. Assemble your work in right order and in the original problem order. (Including the ones that you do not select)

4. You can use the N(0,1) distribution table as attached.

1. Let $Y_1$ and $Y_2$ be independent Bernoulli random variables with parameter $p$. Additionally, let $Z$ be a Bernoulli variable from the same distribution independent of the $Y$s. Let $U_1$ and $U_2$ be independent Bernoulli random variables with parameter $\pi$ independent of the $Y$s and $Z$. Define

$$X_i = (1 - U_i)Y_i + U_i Z$$

for $i = 1, 2$. Let $S = X_1 + X_2$ and $T = U_1 + U_2$.

(a) Find $E(X_i)$ and $Var(X_i)$.

(b) Find $Cov(X_1, X_2)$.

(c) What is the distribution of $T$?

(d) Find the distribution of $S$.

2. Let $X_1, X_2, \ldots, X_n$ be a random sample from a uniform $U(0, \theta)$ distribution. Find the UMVUE of the following

   (a) $\theta$

   (b) $1/\theta$

   (c) $(\theta^{1/2} + \theta^{-1/2})^2$.

3. Let $U_1$ and $U_2$ be two independent random variables from a $U(0,1)$ distribution. Let $X_1 = \sqrt{-2\ln(U_1)}\cos(2\pi U_2)$ and $X_2 = \sqrt{-2\ln(U_1)}\sin(2\pi U_2)$.

   (a) Prove that $X_1$ and $X_2$ are two independent random variables following a standard normal $N(0,1)$ distribution.

   (b) Identify the distribution of $Y = X_2/X_1$.

4. Let $X_1, X_2, \ldots, X_n$ be a random sample from a population with density

$$f(x|\mu, \sigma) = \frac{1}{\sigma} e^{-(x-\mu)/\sigma}, x \geq \mu, -\infty < \mu < \infty, \sigma > 0.$$

(a) Find the maximum likelihood estimates of $\mu$ and $\sigma$.

(b) Find the method of moment estimates of $\mu$ and $\sigma$.

(c) Find the UMVUE of $\mu$ and $\sigma$.

5. (a) Let $T_n$, $n \geq 1$ be a sequence of random variables such that $n^{1/2}(T_n - \theta)$ converges in distribution to $N\big(0, \sigma^2(\theta)\big)$. If $g$ is differentiable, find the asymptotic distribution of $g(T_n)$.

(b) Suppose that $X_1, \ldots, X_n$ is a random sample from $N(\mu, \sigma^2)$. Find $g$ such that the limiting distribution of $g(S_n^2)$ does not depend on $\sigma^2$, where $S_n^2 = \sum_{i=1}^n (X_i - \bar{X}_n)^2/(n-1)$ and $\bar{X}_n = \sum_{i=1}^n X_i/n$.

6. Let $U_1, U_2, \cdots, U_n$ be a random sample of size $n$ from a U(0,1) distribution.

   (a) Identify the probability distribution of $X = -\ln(\prod_{i=1}^{n} U_i)$ with its associated parameter values.

   (b) Using (a), find the pdf of $Y = \prod_{i=1}^{n} U_i$.

   (c) Using (a) and (b), find the asymptotic distribution of $Z = Y^{1/n}$ when $n$ is large.

7. Let $(X, Y, Z)$ be a random vector following a multinomial distribution

$$f(x, y, z) = \frac{n!}{x!y!z!(n-x-y-z)!} p^x q^y r^z (1-p-q-r)^{n-x-y-z},$$

for $x, y, z = 0, 1, 2, \cdots, n$ and $x + y + z \leq n$ where $0 \leq p, q, r, p+q+r \leq 1$.

(a) Find the moment generating function of $X, Y, Z$. That is, find $M(t_1, t_2, t_3) = E(e^{t_1 X + t_2 Y + t_3 Z})$.

(b) Using (a), find the distribution of $X + Y$.

(c) Derive the conditional distribution of $X, Y$ given $Z = z$.

8. Let $X_1, \ldots, X_n$ be iid with pdf

$$f(x|\theta) = \frac{e^{-\theta}\theta^x}{x!} \qquad x = 0, 1, \ldots$$

for $\theta > 0$. Assume a gamma prior on $\theta$,

$$g(\theta) = \frac{1}{\Gamma(r)}\lambda^r \theta^{r-1} e^{-\theta\lambda}$$

(a) Find the posterior distribution of $\theta$.

(b) For a squared-error loss function, find the Bayes estimator of $\theta$.

(c) For a squared-error loss function, find the Bayes estimator of $e^{-\theta}$.

9. Let $X_1, \dots, X_n$ be continuous independent and identically distributed random variables with distribution function $F$. Let $Y = \min\{F(X_1), \dots, F(X_n)\}$ and let $Z = \min[\frac{Y}{\lambda}, \frac{1-Y}{1-\lambda}]$, where $0 < \lambda < 1$.

   (a) Find the distribution function of $Y$.

   (b) Find $P(Z > z)$.

   (c) Hence prove that

$$\lim_{n\to\infty} P\left(Z \leq \frac{1}{n}\right) = 1 - \exp(-\lambda).$$

10. Let $X_1, \ldots, X_m$ be a random sample from the density $\theta_1 x^{\theta_1 - 1}$ for $0 < x < 1$ and let $Y_1, \ldots Y_n$ be a random sample from $\theta_2 x^{\theta_2 - 1}$ for $o < y < 1$. Assume the two samples are independent. Let $U_i = -\log X_i$ for $i = 1, \ldots, m$ and $V_j = -\log Y_j$ for $j = 1, \ldots, n$.

    (a) Find the generalized likelihood-ratio test for testing $H_0 : \theta_1 = \theta_2$ versus $H_1 : \theta_1 \neq \theta_2$.

    (b) Show that the generalized likelihood ratio test can be expressed in terms of the statistic
    $$T = \frac{\sum U_i}{\sum U_i + \sum V_j}$$

    (c) If $H_0$ is true, what is the distribution of $T$?

11. Suppose that $Y_{ij}$ are independent random variables with $Y_{ij} \sim \text{Binomial}(n, p_{ij}), i = 1, 2; j = 1, 2$. Under the model

$$\log\left(\frac{p_{ij}}{1 - p_{ij}}\right) = \mu + \alpha_i + \beta_j,$$

where $\alpha_1 = \beta_1 = 0$,

(a) Find jointly sufficient statistics for $\mu, \alpha_2, \beta_2$.

(b) Write down the log-likelihood function of the unknown parameters $\mu, \alpha_2, \beta_2$.

(c) Write down the expression for the Fisher Information matrix.

(d) Discuss how you would go about computing the MLE's of $\mu, \alpha_2, \beta_2$.

(e) If you collected data, and from the data you got MLE's $\hat{\alpha}_2 = -.9$; (standard error $= 0.06$); and $\hat{\beta}_2 = -4.5$ (standard error $= 0.08$),how would you compare the binomial populations?

12. Let $X_1, \dots, X_m$ be a random sample from an exponential distribution with density

$$f_X(x) = \mu\lambda \exp(-\mu\lambda x), \quad x > 0$$

and let $Y_1, \dots, Y_n$ be a random sample from an exponential distribution with density

$$f_Y(y) = \lambda \exp(-\lambda y), \quad y > 0.$$

Assume that the $X$'s and $Y$'s are independent. Let $S_X = \sum_{i=1}^{m} X_i$, $S_Y = \sum_{i=1}^{n} Y_i$, $\; R = \frac{S_X}{S_Y}$, and $W = S_Y$.

(a) Write down the joint density of $S_X, S_Y$.

(b) Find the joint density of $R, W$.

(c) Find the marginal density of $R$.

(d) Find the marginal log-likelihood of $\mu$ based on $R$.

(e) Compute the MLE of $\lambda$, when $\mu$ is fixed.

Table of P(Z < z), Z ~N(0,1)

| z | 0.00 | 0.01 | 0.02 | 0.03 | 0.04 | 0.05 | 0.06 | 0.07 | 0.08 | 0.09 |
|---|---|---|---|---|---|---|---|---|---|---|
| **0.0** | 0.50000 | 0.50399 | 0.50798 | 0.51197 | 0.51595 | 0.51994 | 0.52392 | 0.52790 | 0.53188 | 0.53586 |
| **0.1** | 0.53983 | 0.54380 | 0.54776 | 0.55172 | 0.55567 | 0.55962 | 0.56356 | 0.56749 | 0.57142 | 0.57535 |
| **0.2** | 0.57926 | 0.58317 | 0.58706 | 0.59095 | 0.59483 | 0.59871 | 0.60257 | 0.60642 | 0.61026 | 0.61409 |
| **0.3** | 0.61791 | 0.62172 | 0.62552 | 0.62930 | 0.63307 | 0.63683 | 0.64058 | 0.64431 | 0.64803 | 0.65173 |
| **0.4** | 0.65542 | 0.65910 | 0.66276 | 0.66640 | 0.67003 | 0.67364 | 0.67724 | 0.68082 | 0.68439 | 0.68793 |
| **0.5** | 0.69146 | 0.69497 | 0.69847 | 0.70194 | 0.70540 | 0.70884 | 0.71226 | 0.71566 | 0.71904 | 0.72240 |
| **0.6** | 0.72575 | 0.72907 | 0.73237 | 0.73565 | 0.73891 | 0.74215 | 0.74537 | 0.74857 | 0.75175 | 0.75490 |
| **0.7** | 0.75804 | 0.76115 | 0.76424 | 0.76730 | 0.77035 | 0.77337 | 0.77637 | 0.77935 | 0.78230 | 0.78524 |
| **0.8** | 0.78814 | 0.79103 | 0.79389 | 0.79673 | 0.79955 | 0.80234 | 0.80511 | 0.80785 | 0.81057 | 0.81327 |
| **0.9** | 0.81594 | 0.81859 | 0.82121 | 0.82381 | 0.82639 | 0.82894 | 0.83147 | 0.83398 | 0.83646 | 0.83891 |
| **1.0** | 0.84134 | 0.84375 | 0.84614 | 0.84849 | 0.85083 | 0.85314 | 0.85543 | 0.85769 | 0.85993 | 0.86214 |
| **1.1** | 0.86433 | 0.86650 | 0.86864 | 0.87076 | 0.87286 | 0.87493 | 0.87698 | 0.87900 | 0.88100 | 0.88298 |
| **1.2** | 0.88493 | 0.88686 | 0.88877 | 0.89065 | 0.89251 | 0.89435 | 0.89617 | 0.89796 | 0.89973 | 0.90147 |
| **1.3** | 0.90320 | 0.90490 | 0.90658 | 0.90824 | 0.90988 | 0.91149 | 0.91309 | 0.91466 | 0.91621 | 0.91774 |
| **1.4** | 0.91924 | 0.92073 | 0.92220 | 0.92364 | 0.92507 | 0.92647 | 0.92785 | 0.92922 | 0.93056 | 0.93189 |
| **1.5** | 0.93319 | 0.93448 | 0.93574 | 0.93699 | 0.93822 | 0.93943 | 0.94062 | 0.94179 | 0.94295 | 0.94408 |
| **1.6** | 0.94520 | 0.94630 | 0.94738 | 0.94845 | 0.94950 | 0.95053 | 0.95154 | 0.95254 | 0.95352 | 0.95449 |
| **1.7** | 0.95543 | 0.95637 | 0.95728 | 0.95818 | 0.95907 | 0.95994 | 0.96080 | 0.96164 | 0.96246 | 0.96327 |
| **1.8** | 0.96407 | 0.96485 | 0.96562 | 0.96638 | 0.96712 | 0.96784 | 0.96856 | 0.96926 | 0.96995 | 0.97062 |
| **1.9** | 0.97128 | 0.97193 | 0.97257 | 0.97320 | 0.97381 | 0.97441 | 0.97500 | 0.97558 | 0.97615 | 0.97670 |
| **2.0** | 0.97725 | 0.97778 | 0.97831 | 0.97882 | 0.97932 | 0.97982 | 0.98030 | 0.98077 | 0.98124 | 0.98169 |
| **2.1** | 0.98214 | 0.98257 | 0.98300 | 0.98341 | 0.98382 | 0.98422 | 0.98461 | 0.98500 | 0.98537 | 0.98574 |
| **2.2** | 0.98610 | 0.98645 | 0.98679 | 0.98713 | 0.98745 | 0.98778 | 0.98809 | 0.98840 | 0.98870 | 0.98899 |
| **2.3** | 0.98928 | 0.98956 | 0.98983 | 0.99010 | 0.99036 | 0.99061 | 0.99086 | 0.99111 | 0.99134 | 0.99158 |
| **2.4** | 0.99180 | 0.99202 | 0.99224 | 0.99245 | 0.99266 | 0.99286 | 0.99305 | 0.99324 | 0.99343 | 0.99361 |
| **2.5** | 0.99379 | 0.99396 | 0.99413 | 0.99430 | 0.99446 | 0.99461 | 0.99477 | 0.99492 | 0.99506 | 0.99520 |
| **2.6** | 0.99534 | 0.99547 | 0.99560 | 0.99573 | 0.99585 | 0.99598 | 0.99609 | 0.99621 | 0.99632 | 0.99643 |
| **2.7** | 0.99653 | 0.99664 | 0.99674 | 0.99683 | 0.99693 | 0.99702 | 0.99711 | 0.99720 | 0.99728 | 0.99736 |
| **2.8** | 0.99744 | 0.99752 | 0.99760 | 0.99767 | 0.99774 | 0.99781 | 0.99788 | 0.99795 | 0.99801 | 0.99807 |
| **2.9** | 0.99813 | 0.99819 | 0.99825 | 0.99831 | 0.99836 | 0.99841 | 0.99846 | 0.99851 | 0.99856 | 0.99861 |
| **3.0** | 0.99865 | 0.99869 | 0.99874 | 0.99878 | 0.99882 | 0.99886 | 0.99889 | 0.99893 | 0.99896 | 0.99900 |
| **3.1** | 0.99903 | 0.99906 | 0.99910 | 0.99913 | 0.99916 | 0.99918 | 0.99921 | 0.99924 | 0.99926 | 0.99929 |
| **3.2** | 0.99931 | 0.99934 | 0.99936 | 0.99938 | 0.99940 | 0.99942 | 0.99944 | 0.99946 | 0.99948 | 0.99950 |
| **3.3** | 0.99952 | 0.99953 | 0.99955 | 0.99957 | 0.99958 | 0.99960 | 0.99961 | 0.99962 | 0.99964 | 0.99965 |
| **3.4** | 0.99966 | 0.99968 | 0.99969 | 0.99970 | 0.99971 | 0.99972 | 0.99973 | 0.99974 | 0.99975 | 0.99976 |
| **3.5** | 0.99977 | 0.99978 | 0.99978 | 0.99979 | 0.99980 | 0.99981 | 0.99981 | 0.99982 | 0.99983 | 0.99983 |
| **3.6** | 0.99984 | 0.99985 | 0.99985 | 0.99986 | 0.99986 | 0.99987 | 0.99987 | 0.99988 | 0.99988 | 0.99989 |
| **3.7** | 0.99989 | 0.99990 | 0.99990 | 0.99990 | 0.99991 | 0.99991 | 0.99992 | 0.99992 | 0.99992 | 0.99992 |
| **3.8** | 0.99993 | 0.99993 | 0.99993 | 0.99994 | 0.99994 | 0.99994 | 0.99994 | 0.99995 | 0.99995 | 0.99995 |
| **3.9** | 0.99995 | 0.99995 | 0.99996 | 0.99996 | 0.99996 | 0.99996 | 0.99996 | 0.99996 | 0.99997 | 0.99997 |
| **4.0** | 0.99997 | 0.99997 | 0.99997 | 0.99997 | 0.99997 | 0.99997 | 0.99998 | 0.99998 | 0.99998 | 0.99998 |
| **4.1** | 0.99998 | 0.99998 | 0.99998 | 0.99998 | 0.99998 | 0.99998 | 0.99998 | 0.99998 | 0.99999 | 0.99999 |