Statistics Ph.D. Qualifying Exam: Part I

November 5, 2011

Student Name: ____

1. Answer 8 out of 12 problems. Mark the problems you selected in the following table.

Problem	1	2	3	4	5	6	7	8	9	10	11	12
Selected												
Scores												

2. Write your answer right after each problem selected, attach more pages if necessary. **Do not** write your answers on the back.

3. Assemble your work in right order and in the original problem order. (Including the ones that you do not select)

4. You can use the N(0,1) distribution table as attached.

1. Suppose that a person plays a game in which he draws a ball from a box of 10 balls numbered 0 through 9. He then puts the ball back and continue to draw a ball (with replacement) until he draws another number which is equal or higher than the first draw. Let X denote the number drawn in the first draw and Y denote the number of subsequent draws needed.

   (a) Find the conditional probability distribution of Y given $X = x$, for $x = 0, 1, \cdots, 9$.

   (b) Find the joint probability distribution of X and Y, $P(X = x, Y = y)$.

   (c) Find the probability distribution of Y, $P(Y = y)$.

   (d) Find $E(Y)$.

2. Let X_1 and X_2 be two independent random variables following a $U(0,1)$ distribution. For each of the Y defined below, identify its distribution with its associated parameter values (no derivation of its probability density function is necessary):

   (a) $Y = -\ln(X_1) - \ln(X_2)$.

   (b) $Y = \ln(X_1) - \ln(X_2)$.

   (c) $Y = \ln(X_1)/\ln(X_2)$.

3. Let $X|R \sim Poisson(\lambda R)$, and $Y|R \sim Poisson(\lambda(1-R))$. Assume that conditional on R, X and Y are independent. If $R \sim Beta(\alpha, \beta)$,

   (a) Find $P(X = k|X + Y = n)$.

   (b) Find $E(XY)$.

   (c) Hence find $Cov(X, Y)$.

4. Let $Y_1, \ldots, Y_n$ be independent random variables, such that $Y_i \sim Poisson(a_i\beta)$, where $a_i > 0, i = 1, \ldots, n$ and $\sum_{i=1}^{n} a_i = 1$.

   (a) Find a complete sufficient statistic T for β.

   (b) Find the MLE of e^{β}.

   (c) Suppose that β has a $Gamma(\beta_0\alpha, \alpha)$, and using a squared error loss function, find the Bayes estimator for β.

   (d) Is the Bayes Estimator a function of T?

5. Let $X_1, \ldots, X_n$ be a random sample from $N(\mu, \sigma_1^2)$ and let $Y_1, \ldots, Y_m$ be a random sample from $N(\mu, \sigma_2^2)$, where the X's and the Y's are independent. For testing

$$H_0 : \sigma_1^2 = \sigma_2^2 \quad \text{versus} \quad H_1 : \sigma_1^2 \neq \sigma_2^2,$$

construct a level α likelihood ratio test, in test, giving the critical values in terms of a standard distribution. Please give all the details of your derivation.

6. The trinomial distribution of two random variables X, Y is given by

$$f_{XY}(x,y) = \frac{n!}{x!y!(n-x-y)!} p^x q^y (1-p-q)^{n-x-y} \qquad x, y = 0, 1, \ldots, n \quad x+y \leq n$$

(a) What is the marginal distribution of X?

(b) Find the conditional distribution of X given Y and obtain $E(X|Y)$.

(c) Compute $Cov(X, Y)$

7. Let B and C be two iid random variables with $U(0, 2)$ distribution. Find the probability that $x^2 + 2Bx + C = 0$ has two real roots.

8. Let $X_1, X_2, \cdots, X_n$ be a random sample from a population with probability function

$$f(x; |\theta) = 2\theta^{-1} x e^{(-x^2/\theta)}, \quad x > 0$$

where $\theta > 0$ is the unknown parameter.

(a) Find the UMVUE for θ.

(b) Find the UMVUE for θ^2.

(c) Find the Cramer-Rao lower bound for unbiased estimators of θ^2.

9. Suppose that $X_1, X_2, \cdots, X_n$ is a random sample of size n from a Normal distribution with mean θ and variance 1, i.e., $N(\theta, 1)$, and we wish to test the hypotheses, $H_0 : \theta = \theta_0$ against the alternative $H_1 : \theta \neq \theta_0$. Let

$$L(x_1, x_2, \cdots, x_n; \theta) = \prod_{i=1}^{n} f(x_i; \theta)$$

be the likelihood function of θ. The likelihood ration test is based on the statistic

$$\lambda = \frac{L(x_1, x_2, \cdots, x_n; \theta_0)}{L(x_1, x_2, \cdots, x_n; \hat{\theta}_m)}$$

where $\hat{\theta}_m$ is the maximum likelihood estimate of θ.

(a) Find the test statistic λ.

(b) Find the distribution of $-2\ln(\lambda)$ under $H_0 : \theta = \theta_0$.

10. Let $X_1, X_2, ..., X_n$ be i.i.d. random variables with p.d.f.

$$f(x, \alpha, \beta) = \begin{cases} \dfrac{1}{2\beta}, & \alpha - \beta \leq x \leq \alpha + \beta \\ 0 & \text{, elsewhere,} \end{cases}$$

where $\beta > 0$ and $-\infty < \alpha < \infty$ are unknown parameters.

(a) Find the method of moment estimates of α and β.

(b) Find the UMVUE for α and β.

11. A more general form of the p.d.f. (probability density function) of a Gamma Distribution is given by

$$f(x; \mu, k) = \frac{1}{\Gamma(k)}(k/\mu)^k x^{k-1} e^{-kx/\mu}, x > 0, \mu, k > 0.$$

Suppose that $X_1, X_2, \cdots, X_n$ are independent and identical exponential random variables with mean μ and $\bar{X} = \sum_{i=1}^{n} X_i/n$.

(a) Show that $\bar{X}$ has gamma distribution with p.d.f. given above with $k = n$.

(b) Find the mean and variance of $\bar{X}$.

(c) Show that $\ln(\bar{X})$ is approximately normal with mean $= \ln \mu$ and variance $1/n$.

12. Let $X_{(1)} < \ldots < X_{(n)}$ be the order statistics of a sample of size n from an exponential distribution with parameter $\lambda = 1$. Show that

$$nX_{(1)}, (n-1)(X_{(2)} - X_{(1)}), (n-2)(X_{(3)} - X_{(2)}), \ldots, (X_{(n)} - X_{(n-1)})$$

are iid exponential random variables with $\lambda = 1$.

Table of P(Z < z), Z ~N(0,1)

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.50000	0.50399	0.50798	0.51197	0.51595	0.51994	0.52392	0.52790	0.53188	0.53586
0.1	0.53983	0.54380	0.54776	0.55172	0.55567	0.55962	0.56356	0.56749	0.57142	0.57535
0.2	0.57926	0.58317	0.58706	0.59095	0.59483	0.59871	0.60257	0.60642	0.61026	0.61409
0.3	0.61791	0.62172	0.62552	0.62930	0.63307	0.63683	0.64058	0.64431	0.64803	0.65173
0.4	0.65542	0.65910	0.66276	0.66640	0.67003	0.67364	0.67724	0.68082	0.68439	0.68793
0.5	0.69146	0.69497	0.69847	0.70194	0.70540	0.70884	0.71226	0.71566	0.71904	0.72240
0.6	0.72575	0.72907	0.73237	0.73565	0.73891	0.74215	0.74537	0.74857	0.75175	0.75490
0.7	0.75804	0.76115	0.76424	0.76730	0.77035	0.77337	0.77637	0.77935	0.78230	0.78524
0.8	0.78814	0.79103	0.79389	0.79673	0.79955	0.80234	0.80511	0.80785	0.81057	0.81327
0.9	0.81594	0.81859	0.82121	0.82381	0.82639	0.82894	0.83147	0.83398	0.83646	0.83891
1.0	0.84134	0.84375	0.84614	0.84849	0.85083	0.85314	0.85543	0.85769	0.85993	0.86214
1.1	0.86433	0.86650	0.86864	0.87076	0.87286	0.87493	0.87698	0.87900	0.88100	0.88298
1.2	0.88493	0.88686	0.88877	0.89065	0.89251	0.89435	0.89617	0.89796	0.89973	0.90147
1.3	0.90320	0.90490	0.90658	0.90824	0.90988	0.91149	0.91309	0.91466	0.91621	0.91774
1.4	0.91924	0.92073	0.92220	0.92364	0.92507	0.92647	0.92785	0.92922	0.93056	0.93189
1.5	0.93319	0.93448	0.93574	0.93699	0.93822	0.93943	0.94062	0.94179	0.94295	0.94408
1.6	0.94520	0.94630	0.94738	0.94845	0.94950	0.95053	0.95154	0.95254	0.95352	0.95449
1.7	0.95543	0.95637	0.95728	0.95818	0.95907	0.95994	0.96080	0.96164	0.96246	0.96327
1.8	0.96407	0.96485	0.96562	0.96638	0.96712	0.96784	0.96856	0.96926	0.96995	0.97062
1.9	0.97128	0.97193	0.97257	0.97320	0.97381	0.97441	0.97500	0.97558	0.97615	0.97670
2.0	0.97725	0.97778	0.97831	0.97882	0.97932	0.97982	0.98030	0.98077	0.98124	0.98169
2.1	0.98214	0.98257	0.98300	0.98341	0.98382	0.98422	0.98461	0.98500	0.98537	0.98574
2.2	0.98610	0.98645	0.98679	0.98713	0.98745	0.98778	0.98809	0.98840	0.98870	0.98899
2.3	0.98928	0.98956	0.98983	0.99010	0.99036	0.99061	0.99086	0.99111	0.99134	0.99158
2.4	0.99180	0.99202	0.99224	0.99245	0.99266	0.99286	0.99305	0.99324	0.99343	0.99361
2.5	0.99379	0.99396	0.99413	0.99430	0.99446	0.99461	0.99477	0.99492	0.99506	0.99520
2.6	0.99534	0.99547	0.99560	0.99573	0.99585	0.99598	0.99609	0.99621	0.99632	0.99643
2.7	0.99653	0.99664	0.99674	0.99683	0.99693	0.99702	0.99711	0.99720	0.99728	0.99736
2.8	0.99744	0.99752	0.99760	0.99767	0.99774	0.99781	0.99788	0.99795	0.99801	0.99807
2.9	0.99813	0.99819	0.99825	0.99831	0.99836	0.99841	0.99846	0.99851	0.99856	0.99861
3.0	0.99865	0.99869	0.99874	0.99878	0.99882	0.99886	0.99889	0.99893	0.99896	0.99900
3.1	0.99903	0.99906	0.99910	0.99913	0.99916	0.99918	0.99921	0.99924	0.99926	0.99929
3.2	0.99931	0.99934	0.99936	0.99938	0.99940	0.99942	0.99944	0.99946	0.99948	0.99950
3.3	0.99952	0.99953	0.99955	0.99957	0.99958	0.99960	0.99961	0.99962	0.99964	0.99965
3.4	0.99966	0.99968	0.99969	0.99970	0.99971	0.99972	0.99973	0.99974	0.99975	0.99976
3.5	0.99977	0.99978	0.99978	0.99979	0.99980	0.99981	0.99981	0.99982	0.99983	0.99983
3.6	0.99984	0.99985	0.99985	0.99986	0.99986	0.99987	0.99987	0.99988	0.99988	0.99989
3.7	0.99989	0.99990	0.99990	0.99990	0.99991	0.99991	0.99992	0.99992	0.99992	0.99992
3.8	0.99993	0.99993	0.99993	0.99994	0.99994	0.99994	0.99994	0.99995	0.99995	0.99995
3.9	0.99995	0.99995	0.99996	0.99996	0.99996	0.99996	0.99996	0.99996	0.99997	0.99997
4.0	0.99997	0.99997	0.99997	0.99997	0.99997	0.99997	0.99998	0.99998	0.99998	0.99998
4.1	0.99998	0.99998	0.99998	0.99998	0.99998	0.99998	0.99998	0.99998	0.99999	0.99999