Statistics Ph.D. Qualifying Exam: Part I

October 18, 2003

Student Name: ______________________________

1. Answer 8 out of 12 problems. Mark the problems you selected in the following table.

1	2	3	4	5	6	7	8	9	10	11	12

2. Write your answer right after each problem selected, attach more pages if necessary.

3. Assemble your work in right order and in the original problem order.

1. Let $\{X_1, \ldots, X_n\}$ be a random sample from the probability distribution with density $f(x, \theta, \phi) = \frac{1}{\phi} e^{-\frac{1}{\phi}(x-\theta)}, x \geq \theta$, where $\phi > 0$.

   (a) Derive the joint sufficient and complete statistics for (θ, ϕ).

   (b) Derive the UMVUE (Uniformly Minimum Variance Unbiased Estimator) of θ.

   (c) What is the UMVUE of ϕ?

2. Let $\{X_1, \ldots, X_n\}$ be a random sample from the Poisson distribution with density $f(x, \theta) = e^{-\theta}\,\frac{\theta^x}{x!}, x = 0, 1, \ldots,$ where $\theta > 0$. Let $\phi(\theta) = P(X = 0) + P(X = 1)$.

   (a) Derive the UMVUE (Uniformly Minimum Variance Unbiased Estimator) of $\phi(\theta)$.

   (b) Assume that the density function of the prior distribution is given by $h(\theta) \propto \theta^{a-1}e^{-b\theta}$, where $a > 0$ and $b > 0$ are known. Derive the Bayes estimator of $\phi(\theta)$ under the loss function $l(\theta, s) = (s - \theta)^2$ for the estimator s of θ.

3. Let $(X_1, Y_1), \ldots, (X_n, Y_n)$ be independently and identically distributed as (X, Y), where (X, Y) follows a bivariate normal distribution with means $EX = \mu_1$ and $EY = \mu_2$ and with variances and covariance as $Var(X) = \sigma_1^2$, $Var(Y) = \sigma_2^2$ and $Cov(X, Y) = \rho\sigma_1\sigma_2$, respectively. Put $Z_i = X_i - Y_i, i = 1, \ldots, n$.

   (a) Based on the observed Z values, derive the likelihood ratio test (LRT) for testing $H_0 : \mu_1 = \mu_2$ against the alternative hypothesis $H_1 : \mu_1 \neq \mu_2$.

   (b) Derive the probability distribution of your test statistic under H_0.

   (c) What is the p-value of the LRT test?

4. Let X, Y, U be independent random variables with $X \sim \text{Poisson}(\lambda)$, $Y \sim \text{Poisson}(\mu)$, and $U \sim \text{Uniform}(0, 1)$ Let

$$V = \begin{cases} X & , \quad \text{if} \quad U > a \\ Y & , \quad \text{if} \quad U \leq a, \end{cases}$$

Find

$$P(X + Y = n \mid V = k)$$

5. Suppose that T is a sufficient statistic for unknown parameter θ, based on a random sample $\mathbf{X}$.

   (a) Prove that the Bayes estimator $d_B(\mathbf{X})$ of θ is a function of T.

   (b) Under what conditions can you construct a UMVUE estimate of θ which is a function of $d_B(\mathbf{X})$? (Justify your answer fully.)

6. Let $\{X_1, \dots, X_n\}$ be independently and identically distributed with density $f(x; \mu, \sigma^2)$, where μ is the mean value and σ^2 the variance. Put $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$ and $S^2 = \sum_{i=1}^{n}(X_i - \bar{X})^2$.

   (a) If $f(x; \mu, \sigma^2)$ is a normal density, show that $\bar{X}$ and S^2 are independently distributed of each other.

   (b) If $\bar{X}$ and S^2 are independently distributed of each other, what can you say about the density $f(x; \mu, \sigma^2)$?

7. A more general form of the p.d.f. (probability density function) of a Gamma Distribution is given by

$$f(x; \mu, k) = \frac{1}{\Gamma(k)}(k/\mu)^k x^{k-1} e^{-kx/\mu}, x > 0, \mu, k > 0.$$

Suppose that $X_1, X_2, \cdots, X_n$ are independent and identical exponential random variables with mean μ and $\bar{X} = \sum_{i=1}^{n} /n$.

(a) Show that $\bar{X}$ has gamma distribution with p.d.f. given above with $k = n$.

(b) Find the mean and variance of $\bar{X}$.

(c) Show that $\ln \bar{X}$ is approximately normal with mean $= \ln \mu$ and variance $1/n$.

8. Let $Y_1 < Y_2 < \cdots < Y_n$ be the order statistics of n independent observations from a $U(0, \theta)$ distribution.

   (a) Show that Y_n is a complete sufficient statistics for θ and then prove its p.d.f. is

   $$g(y_n; \theta) = n y_n^{n-1} / \theta^n, \quad 0 < y_n < \theta$$

   and zero elsewhere.

   (b) Find the distribution function $F_n(z; \theta)$ of $Z_n = n(\theta - Y_n)$.

   (c) Find the $\lim_{n \to \infty} F_n(z; \theta)$ and thus the limiting distribution of Z_n.

9. Suppose that $X_1, X_2, \cdots, X_n$ is a random sample of size n from a Normal distribution with mean θ and variance 1, i.e., $N(\theta, 1)$, and we wish to test the hypotheses, $H_0 : \theta = \theta_0$ against the alternative $H_1 : \theta \neq \theta_0$. Let

$$L(x_1, x_2, \cdots, x_n; \theta) = \prod_{i=1}^{n} f(x_i; \theta)$$

be the likelihood function of θ. The likelihood ration test is based on the statistic

$$\lambda = \frac{L(x_1, x_2, \cdots, x_n; \theta_0)}{L(x_1, x_2, \cdots, x_n; \hat{\theta}_m)}$$

where $\hat{\theta}_m$ is the maximum likelihood estimate of θ.

(a) Find the test statistic λ.

(b) Find the distribution of $-2\ln(\lambda)$ under $H_0 : \theta = \theta_0$.

10. Let X and Y be two random variables. Prove or disprove that

$$Var[X] \geq Var[E(X|Y)] \geq Var[E(X|Y^2)].$$

11. Let $X_1, \dots, X_n$ be i.i.d. from the uniform distribution on the interval $(\theta, \theta + 1)$.

(a) Find the joint distribution of $X_{(1)}$ and $X_{(n)}$.

(b) Explain how to find MLE for the parameter θ.

(c) Find a UMP test of size α for testing $H_0 : \theta \leq 0$ versus $H_1 : \theta > 0$.

12. Let X_1, X_2 and X_3 be independently distributed central chi-square variables with degrees of freedoms f_1, f_2 and f_3 respectively. Put $Y_1 = \frac{f_3 X_1}{f_1 X_3}$ and $Y_2 = \frac{f_3 X_2}{f_2 X_3}$.

   (a) Obtain the joint pdf (probability density function) of $\{Y_1, Y_2\}$. (Be sure to give the support of $\{Y_1, Y_2\}$.)

   (b) Express the probability $P\{Y_1 < 2Y_2\}$ in terms of an incomplete Beta-integral. (The incomplete Beta-integral $I_a(p, q)$ is defined by $I_a(p, q) = \frac{1}{B(p,q)} \int_0^a x^{p-1}(1 - x)^{q-1} dx$ for $\{p > 0, q > 0, 0 < a < 1\}$.)