Ph.D. Qualifying Exam
Real Analysis
September 5, 2009

Complete any **five** of the following **nine** problems. If you do more than five problems, only the top five will count towards your grade. You have three hours.

1. (a) State carefully and precisely the Fundamental Theorem of Calculus for the Lebesgue integral.

(b) Give an example of a function which is of bounded variation on $[0,1]$ but is not continuous on $[0,1]$. Can such a function be the indefinite integral of a Lebesgue integrable function? Give your reasons.

(c) Consider the function

$$f(x) = \begin{cases} x^{\alpha}\cos\left(\frac{1}{x}\right) & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}.$$

For which values of α is f absolutely continuous on $[0,1]$?

2. Let $(\Omega, \mathcal{A}) = (\mathcal{N}, \mathcal{P}(\mathcal{N}))$ be a measurable space. Here $\mathcal{N}$ is the set of positive integers, and $\mathcal{P}(\mathcal{N})$ is the power set of $\mathcal{N}$. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers with $\sum_{n=1}^{\infty} |a_n| < \infty$. Define a measure ν on $\mathcal{P}(\mathcal{N})$ by

$$\nu(A) = \sum_{n \in A} a_n$$

for each $A \in \mathcal{A}$.

(a) Prove that ν is a signed measure.

(b) Determine ν^{+}, ν^{-}, and $|\nu|$.

3. (a) State and prove the Monotone Convergence Theorem.

(b) Let g be a nonnegative, measurable function on $(-\infty, \infty)$. Show that

$$\lim_{n\to\infty} \int_{-n}^{n} \frac{n}{x^2+n} g(x)\, dx = \int_{-\infty}^{\infty} g(x)\, dx.$$

4. (a) State carefully and precisely the Lebesgue Dominated Convergence Theorem.

(b) Let $(\mathbb{R}, \mathcal{M}, \lambda)$ be the measure space on $\mathbb{R}$ where $\mathcal{M}$ represents the Borel sets on $\mathbb{R}$ and λ is Lebesgue measure on $\mathbb{R}$. The Fourier transform of f is denoted by $\widehat{f}$ and is defined by

$$\widehat{f}(t) = \int_{-\infty}^{\infty} e^{-itx} f(x)\, d\lambda(x)$$

for $t \in \mathbb{R}$ and $f \in L^1(\mathbb{R})$. (Note that $d\lambda(x) = dx$ since λ is Lebesgue measure.)

(i) Prove that $\widehat{f}$ is continuous on $\mathbb{R}$.

(ii) Prove that if $\int_{-\infty}^{\infty} |xf(x)|\, d\lambda(x) < \infty$, then $\widehat{f}$ is differentiable on $\mathbb{R}$, and

$$\left(\widehat{f}\right)'(t) = \int_{-\infty}^{\infty} (-ix)\, e^{-itx} f(x)\ d\lambda(x)$$

for $t \in \mathbb{R}$.

5. (a) State the Radon-Nikodym Theorem.

(b) Let μ and ν be measures on the same $\sigma-$finite measurable space $(\Omega, \mathcal{A})$. Suppose that μ and ν satisfy the property that for any measurable set $A \in \mathcal{A}$, $\mu(A) = 0$ if $\nu(A) = 0$. Prove that for each $\epsilon > 0$, there is a $\delta > 0$ such that $\mu(A) < \epsilon$ for any measureable set A with $\nu(A) < \delta$.

6. (a) State the definition of a normed space.

(b) State the definition of a Banach space.

(c) Let $C[0,1]$ be the space of continuous functions on the interval $[0,1]$. If $f \in C[0,1]$, define

$$\|f\|_p = \left(\int_0^1 |f(x)|^p\ dx\right)^{\frac{1}{p}}.$$

Show that $C[0,1]$ equiped with the norm $\|\cdot\|_p$ for $1 \leq p < \infty$ is a normed space, but it is not a Banach space.

7. Let $r, s \in [1, \infty]$ and let $(\Omega, \mathcal{A}, \mu)$ be a measure space.

(a) Suppose $p \in [1, \infty]$ such that

$$\frac{1}{p} = \frac{1}{r} + \frac{1}{s}.$$

Show that if $f \in L^r(\Omega, \mathcal{A}, \mu)$ and $g \in L^s(\Omega, \mathcal{A}, \mu)$, then $fg \in L^p(\Omega, \mathcal{A}, \mu)$ and

$$\|fg\|_p \leq \|f\|_r \|g\|_s.$$

(b) Now suppose $(\Omega, \mathcal{A}, \mu)$ is a finite measure space. Show that if $1 \leq s < r \leq \infty$,

$$L^r(\Omega, \mathcal{A}, \mu) \subseteq L^s(\Omega, \mathcal{A}, \mu).$$

(c) Show by example that the containment in (b) fails if $\Omega = (0, \infty)$ where μ is Lebesgue measure.

8. Let

$$f(x,y) = \begin{cases} \frac{xy}{(x^2+y^2)^2} & \text{if} \quad x^2 + y^2 > 0 \\ 0 & \text{if} \quad x = y = 0 \end{cases}.$$

Is f integrable on $\mathbb{R}^2$? Why or why not?

9. (a) State the Hahn-Banach theorem.
(b) Let

$$\ell^1 = \left\{ x = (x_n)_{n \geq 1} : x_n \in \mathbb{C} \text{ for } n = 1, 2, ..\text{and } \sum_{n=1}^{\infty} |x_n| < \infty \right\}$$

. Equip ℓ^1 with the norm

$$\|x\|_1 = \sum_{n=1}^{\infty} |x_n| \, .$$

Construct a bounded linear functional on some subspace M of ℓ^1 which has two (hence infinitely many) distinct norm-preserving extensions to ℓ^1.

(c) In (b) , if the Banach space ℓ^1 is replaced by the Hilbert space ℓ^2,

$$\ell^2 = \left\{ x = (x_n)_{n \geq 1} : x_n \in \mathbb{C} \text{ for } n = 1, 2, ..\text{and } \sum_{n=1}^{\infty} |x_n|^2 < \infty \right\},$$

with the inner product

$$\langle x, y \rangle = \sum_{n=1}^{\infty} x_n \overline{y_n},$$

then the extension is unique. What is this extension? (You may need to modify M so that $M \subseteq \ell^2$.)