PH.D. QUALIFYING EXAM
REAL VARIABLES, 8 SEPTEMBER 2007

You have three hours. Solve any five problems. Credit will be given for the five best solutions. Show work.

Problem 1

(a) Define the outer Lebesgue measure m^*A of $A \subset \mathbb{R}$.
(b) Show that the outer measure m^* is σ-subadditive.
(c) Prove that if $m^*A = 0$, then $m^*(A \cup B) = m^*B$.

Problem 2

Let f be a positive (a.e.) Lebesgue measurable function on $\mathbb{R}$, and let for $i = 0, \pm 1, \pm 2, \ldots$,

$$a_i = m\{x \in \mathbb{R} : 2^{i-1} < f(x) \leq 2^i\},$$

where m is the Lebesgue measure on $\mathbb{R}$. Show that f is integrable over $\mathbb{R}$ if and only if $\sum_{i=-\infty}^{\infty} 2^i a_i < \infty$.

Problem 3

Show that if f is absolutely continuous on $[a, b]$, then it is of bounded variation on $[a, b]$.

Problem 4

Show that the Lebesgue space L^p, $1 \leq p < \infty$, is complete. Here L^p is considered over the measure space $([0, 1], \mathcal{M}, m)$.

Problem 5

Suppose (x_n) is a weakly convergent sequence in a Banach space $\mathcal{B}$.
(a) Show that $\|x_n\|$ is bounded.
(b) Suppose $K \subset \mathcal{B}$ is a compact set and $x_n \in K$ for all n. Show that (x_n) converges strongly and that its weak and strong limits are identical.

Problem 6

On l^p, $1 \leq p < \infty$, consider the operator T, given by

$$T(x_n) = (y_n),$$

where

$$y_k = \frac{1}{k} x_{k+1}, \quad k \in \mathbb{N}.$$

(a) Show that T is a bounded linear operator. Determine its norm.
(b) Show that T is neither surjective nor injective. Prove that its range $R(T)$ is dense in l^p.
(c) Let S be the restriction of T to the closed subspace l^p_0 of all sequences (x_n) in l^p such that $x_1 = 0$. Show that S is injective.
(d) Is the inverse of S bounded, closed?

Problem 7

Let $(X, \mathcal{B}, \mu)$ be a complete measure space and suppose that g is integrable with respect to μ. Define the map λ on the σ-algebra $\mathcal{B}$ by

$$\lambda(E) = \int_E g \, d\mu.$$

Prove that λ is a signed measure and that

$$|\lambda|(E) = \int_E |g| \, d\mu \quad \text{for } E \in \mathcal{B}.$$

Problem 8

Suppose $f,\, g \in L^1(\mathbb{R})$. Show that the measurable function

$$F(x, y) = f(x - y)\, g(y)$$

belongs to $L^1(\mathbb{R}^2)$. Conclude that the integral

$$h(x) = \int_{-\infty}^{\infty} f(x - y)\, g(y)\, dy$$

exists for almost all $x \in \mathbb{R}$, that $h \in L^1(\mathbb{R})$, and that

$$\|h\|_1 \leq \|f\|_1 \, \|g\|_1.$$

Note: You need not show that F is measurable.