# Algebra Ph.D. Qualifying Exam

January 2017

Answer any **five** of the following eight questions.
**You should state clearly any general results you use.**

1. Suppose $\sigma$ is an element of order 2 in the alternating group $A_n$. Prove that there exists a $\tau \in S_n$ such that $\tau^2 = \sigma$.

2. Let $G$ be a group of order 56 with a normal 2-Sylow subgroup $Q$, and let $P$ be a 7-Sylow subgroup of $G$. Show that either $G \cong P \times Q$ or $Q \cong C_2 \times C_2 \times C_2$. (Hint: $P$ acts on $Q \setminus \{e\}$ via conjugation; show that this action is either trivial or transitive.)

3. By considering the map $f(X) \mapsto (f(1), f(2))$ or otherwise, prove that $\mathbb{Z}[X]/(X^2 - 3X + 2)$ is isomorphic to the ring $\mathbb{Z} \times \mathbb{Z}$.

4. (a) Show that any finite subgroup of the group of units of an integral domain $R$ is cyclic. (Hint: consider the number of roots of the equation $X^d - 1 = 0$ for suitable $d$.)

   (b) Give an example to show that (a) may fail to hold if $R$ is not an integral domain.

5. Let $p \neq 5$ be an odd prime and let $\zeta$ be a primitive 5th root of 1 in an extension field of $\mathbb{F}_p$. Let $\alpha = \zeta + \zeta^{-1}$. Show that $(2\alpha + 1)^2 = 5$ and that

$$5 \text{ is a square in } \mathbb{F}_p \iff \alpha^p = \alpha \iff p \equiv \pm 1 \bmod 5.$$

6. Let $f(X) = X^4 + 4X^2 + 2$. Find the Galois group of the splitting field of $f$ over the following fields.

   (a) $\mathbb{Q}$. (Hint: show that if $\alpha$ is a root of $f$ then $\sqrt{2}/\alpha$ is also a root.)

   (b) $\mathbb{F}_2$.

   (c) $\mathbb{F}_3$.

7. Let $R$ be an integral domain. Recall that an $R$-module is called *torsion-free* if for every element $a \neq 0$ in $R$ and $m \neq 0$ in the module, $am \neq 0$. Let $N$ be an $R$-submodule of an $R$-module $M$.

   (a) Show that if $N$ and $M/N$ are torsion-free, then so is $M$.

   (b) Show that the converse holds in (a) for every $N$, $M$ if and only if $R$ is a field.

8. Identify with proof $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}[X]$.