# PhD Qualifying Exam: Algebra

September 20, 2008

*Answer any* **five** *of the following eight questions.*
*You should state clearly any general results you use.*

1. (a) Find the number of elements of order five in $D_4 \bigoplus D_{10} \bigoplus D_6$, where $D_{2n}$ represents the dihedral group of order $2n$.

   (b) If $a$ is an element of order $n$ of $D_{2n}$, show that $\langle a \rangle \trianglelefteq D_{2n}$ and $D_{2n}/\langle a \rangle \cong \mathbb{Z}_2$.

2. Determine all non-isomorphic subgroups of order 15. State all the theorems you have used.

3. (a) Determine the center $Z$ of the ring of all $2 \times 2$ matrices $\mathcal{M}_2$ over a field $F$.

   (b) Show that $Z$ is not an ideal in $\mathcal{M}_2$.

   (c) What is the center of the ring of all $n \times n$ matrices over a division ring?

4. Let $G$ be a group, the subgroup of $G$ generated by the set $\{aba^{-1}b^{-1} : a, b \in G\}$ is called the commutator subgroup of $G$, denoted by $G'$. Show that:

   (a) $G' \trianglelefteq G$.

   (b) $G/G'$ is abelian.

   (c) If $N \trianglelefteq G$, then $G/N$ is abelian if and only if $N \geq G'$.

5. (a) Let $G$ be a finite group such that $|G| = p^n$ with $p$ prime. Show that for $k \leq n$, a nonnegative integer, $G$ has a normal subgroup of order $p^k$.

   (b) If $P$ is a normal Sylow $p$-subgroup of a finite group $G$ and $f \colon G \to G$ is an endomorphism, prove that $f(P)$ is a subgroup of $P$.

Please Turn Over.

6. (a) Let $R$ be a commutative ring with 1 and let $M$ be an $R$-module. What does it mean for $M$ to be a free $R$-module?

   (b) Let $\mathbb{Z}[\frac{1}{2}]$ denote the subring of $\mathbb{Q}$ generated by $\mathbb{Z}$ and $\frac{1}{2}$. Prove or disprove: $\mathbb{Z}[\frac{1}{2}]$ is a free $\mathbb{Z}$-module.

7. Let $f(x) = x^5 - 9x + 3$. Determine the Galois group of the splitting field of $f$ over (a) $\mathbb{Q}$, and (b) $\mathbb{F}_2$.

8. Let $K$ be a Galois extension of $\mathbb{Q}$ such that $\mathrm{Gal}(K/\mathbb{Q})$ is a cyclic group of order 12.

   (a) How many intermediate fields are there, and what are their degrees over $\mathbb{Q}$?

   (b) Give an example of such an extension $K$. (Hint: 13 is a prime.)