Ph. D. Qualifying Exam-Algebra (Fall 2005)

1. (a) Let $(G, \circ)$ be a semigroup with two identities e, e'. Show that $e = e'$.

   (b) Let $(R, +, \cdot)$ be a ring. Show that $0 \cdot a = 0$ for all $a \in R$.

   (c) Let $(R, +, \cdot)$ be an integral domain with identity 1. Suppose that there is a natural number n such that $n\,1 = 0$. Show that there is a prime number p such that $p\,a = 0$ for all $a \in R$.

2. Let $(G, \cdot)$ be a group. Do one of the following problems.

   (a) Let H be a subgroup of $Z(G)$, the center of G. Show that if G/H is cyclic, then G is abelian.

   (b) Let G be a group with order 78. Suppose that G has an abelian normal subgroup H with order 6. Show that Sylow 13-subgroup is normal and G is abelian.

3. Do one of the following problems:

   (a) Show that every Euclidean domain is principal ideal domain.

   (b) Let R be a commutative ring and I a proper ideal of R. Show that I is a prime ideal if and only if R/I is an integral domain.

4. Do one of the following problems:

   (a) Let $\mathbb{F}$ be a field with characteristic $p < \infty$. Show that all roots of $x^{p^4} - x \in \mathbb{F}[x]$ are distinct.

   (b) Let $\mathbb{F}$ be the splitting field of $x^3 - 2$ over $\mathbb{Q}$. Find $G(\mathbb{F}/\mathbb{Q})$? What is the order of $G(\mathbb{F}/\mathbb{Q})$?

5. Do one of the following problems:

   (a) Let A be a 3×3 complex matrix such that $A^2(A - I) = 0$. Find all possible nonequivalent Jordan forms for A.

   (b) Let M be a module over a commutative ring R and I be an ideal of R. Show that $\{x \in M : rx = 0 \text{ for all } r \in I\}$ is a submodule of M.