PhD Qualifying Exam: Algebra

March 20th 2004

Answer any **five** *of the following eight questions.*
You should state clearly any general results you use.

1. (a) Let G be a group and K a normal subgroup of G such that $m = [G:K]$ is relatively prime to the order of K. Show that $K = \{x^m : x \in G\}$.
   [Hint: any element of K is in fact of the form x^m for some $x \in K$.]

   (b) Give an example to show that this fails if K is not assumed to to be normal.

2. (a) If G is a finite group, $Z(G) = \{z \in G : zg = gz \text{ for all } g \in G\}$ is the center of G, and $G/Z(G)$ is cyclic, show that G is abelian.

   (b) If G is a group of order p^2n where $p \nmid n$, p prime, and G has a subgroup of order n that is contained in $Z(G)$, show that G is abelian.

   (c) Deduce that if G is a non-abelian group of order p^2q, where p and q are distinct primes, then $Z(G)$ has order either 1 or p.

3. (a) Show that $I = (2, X^2 + 1)$ is not a prime ideal of $\mathbb{Z}[X]$.

   (b) Find two prime ideals P_1 and P_2 of $\mathbb{Z}[X]$ with $0 \neq P_1 \subset I \subset P_2$.

4. Let R be a (not necessarily commutative) ring which contains a field K and assume R is finite dimensional as a vector space over K. If R has no zero divisors, show that R is a division ring (i.e., every non-zero element has a multiplicative inverse).

5. Suppose M/F is a finite Galois extension and K and L are two subfields of M containing F, with K/F Galois. Let KL be the compositum of K and L inside M, i.e., the smallest subfield of M containing both K and L.

   (a) Show that KL/L is Galois.

   (b) Show that $[KL:L]$ divides $[K:F]$.

Turn over

6. Find the Galois group of $X^7 - 1$ over
   (a) $\mathbb{F}_2$,
   (b) $\mathbb{Q}$.

7. Suppose A and B are two $n \times n$ matrices with complex entries with the same minimal polynomials and the same characteristic polynomials.
   (a) If $n = 3$ show that A and B are similar.
   (b) Show that there are non-similar A and B with this property when $n > 3$.

8. Let A be a $\mathbb{Z}$-module and n a positive integer. Show that $(\mathbb{Z}/n\mathbb{Z}) \otimes_{\mathbb{Z}} A$ is isomorphic to A/nA.