QUALIFYING EXAM (SPRING 2002)
ALGEBRA

Answer any **six** of the following eight questions. **You must state clearly any general results you use.**

1. Prove that if G is a non-trivial p-group then the center of G is non-trivial. Deduce that every p-group is solvable.

2. Prove that if G is a simple group of order 168 then G contains exactly 48 elements of order 7.

3. Let R be a ring with 1. Show that if x is contained in every maximal ideal of R then $1 + x$ is a unit.

4. Let R be a commutative ring with 1.

   (a) Show that every maximal ideal is prime.

   (b) Show that if R is a PID then every non-zero prime ideal is maximal.

5. Let $K = \mathbb{Q}(t)$ be the field of rational functions in the indeterminate t. Let $\phi\colon \mathbb{Q}(t) \to \mathbb{Q}(t)$ be the field homomorphism which fixes $\mathbb{Q}$ and sends t to $\frac{2}{t}$.

   (a) Show that ϕ is an automorphism of K of order 2.

   (b) Show that the fixed field of $G = \{1, \phi\}$ is $F = \mathbb{Q}(t + \frac{2}{t})$.

   (c) Find the minimal polynomial of t over F.

6. What is the Galois group of $X^3 + 7X + 7$ over

   (a) $\mathbb{F}_2$ (the field of 2 elements),

   (b) $\mathbb{F}_3$ (the field of 3 elements),

   (c) $\mathbb{Q}$.

   State clearly any results you use.

7. Suppose A and B are finitely generated abelian groups and $A \oplus A \cong B \oplus B$. Show that $A \cong B$.

8. Find the characteristic polynomial, invariant factors, elementary divisors, rational canonical form, and Jordan canonical form of the matrix

$$\begin{pmatrix} -1 & 1 & 1 \\ -2 & 2 & 1 \\ -2 & 1 & 2 \end{pmatrix}$$