QUALIFYING EXAM (FALL 2001)
ALGEBRA

Answer any **six** of the following eight questions. **You must state clearly any general results you use.**

1. Show that the group presentation $\langle a, b \mid a^4 = b^3 = 1,\ ab = ba^3 \rangle$ represents a cyclic group of order 6.

2. If p and q are distinct primes, show that there are no simple groups of order p^2q stating clearly any results that you use.

3. If F is a field, show that the ring of $n \times n$ matrices $M_n(F)$ has no 2-sided ideals other than (0) and $M_n(F)$.

4. Let $d \in \mathbb{Z}$ be a squarefree integer. Show that every non-zero prime ideal of $R = \mathbb{Z}[\sqrt{d}]$ is maximal. [Hint: First show that if P is a non-zero prime ideal then R/P is finite.]

5. Prove that a finite extension K/F is simple if and only if there are only finitely many intermediate fields.

6. What is the Galois group of $X^{12} - 1$ over

   (a) $\mathbb{F}_2$ (the field of 2 elements),

   (b) $\mathbb{F}_5$ (the field of 5 elements),

   (a) $\mathbb{Q}$.

   State clearly any results you use.

7. If A is an abelian group, show that $(\mathbb{Z}/n\mathbb{Z}) \otimes A \cong A/nA$.

8. Let A be an matrix over $\mathbb{C}$ such that $A^n = A$ for some $n > 1$. Show that A is similar to a diagonal matrix.