Statistics Masters Comprehensive Exam

April 12, 2013

Student Name: ______________________________

1. Answer 8 out of 12 problems. Mark the problems you selected in the following table.

Problem	1	2	3	4	5	6	7	8	9	10	11	12
Selected												
Scores												

2. Write your answer right after each problem selected, attach more pages if necessary. **Do not** write your answers on the back.

3. Assemble your work in right order and in the original problem order. (Including the ones that you do not select)

4. You can use the N(0,1) distribution table as attached.

1. Suppose the joint pdf of (X, Y) is given by

$$f_{XY}(x, y) = \begin{cases} 1 - \alpha(1 - 2x)(1 - 2y) & 0 < x < 1 \;\; 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) Find the covariance of X and Y.

(b) If an isosceles triangle is formed with base $2X$ and height Y, where X and Y have the joint density above, what should α be to maximize the expected area of the triangle?

2. Let $X_1, X_2, \ldots, X_n$ be i.i.d. from

$$f(x;\theta) = \frac{\theta}{x^2} \quad x > \theta$$

(a) Find the maximum likelihood estimator of θ. Justify.

(b) Is $X_{(1)} = \min(X_1, \ldots, X_n)$ a sufficient statistic for θ? Explain.

(c) Is the MLE an unbiased estimator of θ? Explain.

3. Let $X_1, \ldots, X_n$ be a random sample from

$$f(x;\theta) = \theta e^{-\theta x} \quad x > 0$$

Find a generalized likelihood ratio test for testing $H_0 : \theta \leq 1$ versus $H_1 : \theta > 1$.

4. It has been documented via numerous research studies that the eldest child in a family with multiple children generally has a higher IQ than his or her siblings. In a certain large population of US families with two children, suppose that the random variable X denotes the IQ of the older child and that the random variable Y denotes the IQ of the younger child. Assume that X and Y have a joint bivariate normal distribution with parameters $E(X) = 110, E(Y) = 100, Var(X) = Var(Y) = 225$, and $\rho = 0.8$.

   (a) Find the probability that the older child has an IQ at least 15 points higher than the younger child.

   (b) If the older child is known to have an IQ of 120, what is the probability that the younger child has IQ greater than 120?

5. Let U be a random variable which is uniformly distributed over the interval (0,1). Define $X = \sin(2\pi U)$ and $Y = \cos(2\pi U)$.

   (a) Are X and Y uncorrelated ? Explain.

   (b) Are X and Y independent ? Explain.

6. Let X be a random variable which has a beta$(\nu_1/2, \nu_2/2)$ distribution over the interval (0,1), where ν_1 and ν_2 are integers. Find a transformation $Y = g(X)$ so that Y has an $F(\nu_1, \nu_2)$ distribution.

7. Bowl C contains six red chips and four blue chips. Two of these 10 chips are selected at random and without replacement and put in bowl D, which was originally empty. One chip is then drawn at random from bowl D.

   (a) Let B_i be the event that i blue chips were transferred from bowl C to bowl D. Find $P(B_i)$ for $i = 0, 1, 2$.

   (b) Given that this chip drawn from bowl D is blue, find the conditional probability that one red chip and one blue chip were transferred from bowl C to bowl D.

8. Let $X_1, X_2, \cdots, X_n$ be a random sample from an Exponential distribution with pdf $f(x;\theta) = 1/\theta e^{-x/\theta}$, $x > 0$. Find the UMVUE of

   (a) θ

   (b) $1/\theta$

9. In order to decide on an appropriate amount to charge as premium, insurance companies often use the exponential principle defined as follows: If X is the amount that it will have to pay in claims to a randomly selected client, then the premium charged by the insurance company should be

$$P = \frac{1}{a} \ln(E[e^{aX}]),$$

where $a > 0$ is a fixed specified constant. Suppose that an insurance company assumes that X has a Poisson distribution with parameter θ.

(a) Find P.

(b) An insurance company wishes to find a maximum likelihood estimator $\hat{P}$ of P, by taking a random sample $X_1, \ldots, X_n$ from a large set of previous payments. Assuming that the X's are a random sample from $\mathcal{P}\text{oisson}(\theta)$, where θ is unknown. Find $\hat{P}$.

(c) Prove that if an insurance company has a practice that fixes a at a very small value, then it will barely break even. (Hint: Let $a \to 0$.)

10. Let $X_1, \ldots, X_n$ be a random sample from a geometric distribution with parameter θ. That is,

$$f(x|\theta) = (1-\theta)^{x-1}\theta, \quad x = 1, 2, 3, \ldots$$

Suppose you put a $\mathcal{B}\text{eta}(\alpha, \beta)$ prior on θ.

(a) Find the posterior distribution of θ.

(b) Find the Bayes estimator of θ, under the loss function $L(\theta, a) = \frac{(\theta-a)^2}{\theta(1-\theta)}$.

(c) Is estimator a function of a complete sufficient statistic for θ?

11. Let X, Y be independent random variables with identical distributions $\mathcal{G}\text{amma}(1, \theta)$. Let $V = X + Y$ and $W = Y/(X + Y)$.

   (a) Find the joint density function of (V, W).

   (b) Find the marginal density of W.

12. Let $X_1, \ldots, X_n$ be a random sample from a $\mathcal{B}\mathrm{eta}(\theta, 1)$.

(a) Construct a uniformly most powerful size α test of $H_0 : \theta \leq 1$ versus $H_1 : \theta > 1$, which is a function of the statistic $T(\mathbf{X}) = \prod_{k=1}^{n} X_k$.

(b) Give an expression for the power function of the test in terms of the percentile of one of the known standard distributions.

Table of P(Z < z), Z ~N(0,1)

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.50000	0.50399	0.50798	0.51197	0.51595	0.51994	0.52392	0.52790	0.53188	0.53586
0.1	0.53983	0.54380	0.54776	0.55172	0.55567	0.55962	0.56356	0.56749	0.57142	0.57535
0.2	0.57926	0.58317	0.58706	0.59095	0.59483	0.59871	0.60257	0.60642	0.61026	0.61409
0.3	0.61791	0.62172	0.62552	0.62930	0.63307	0.63683	0.64058	0.64431	0.64803	0.65173
0.4	0.65542	0.65910	0.66276	0.66640	0.67003	0.67364	0.67724	0.68082	0.68439	0.68793
0.5	0.69146	0.69497	0.69847	0.70194	0.70540	0.70884	0.71226	0.71566	0.71904	0.72240
0.6	0.72575	0.72907	0.73237	0.73565	0.73891	0.74215	0.74537	0.74857	0.75175	0.75490
0.7	0.75804	0.76115	0.76424	0.76730	0.77035	0.77337	0.77637	0.77935	0.78230	0.78524
0.8	0.78814	0.79103	0.79389	0.79673	0.79955	0.80234	0.80511	0.80785	0.81057	0.81327
0.9	0.81594	0.81859	0.82121	0.82381	0.82639	0.82894	0.83147	0.83398	0.83646	0.83891
1.0	0.84134	0.84375	0.84614	0.84849	0.85083	0.85314	0.85543	0.85769	0.85993	0.86214
1.1	0.86433	0.86650	0.86864	0.87076	0.87286	0.87493	0.87698	0.87900	0.88100	0.88298
1.2	0.88493	0.88686	0.88877	0.89065	0.89251	0.89435	0.89617	0.89796	0.89973	0.90147
1.3	0.90320	0.90490	0.90658	0.90824	0.90988	0.91149	0.91309	0.91466	0.91621	0.91774
1.4	0.91924	0.92073	0.92220	0.92364	0.92507	0.92647	0.92785	0.92922	0.93056	0.93189
1.5	0.93319	0.93448	0.93574	0.93699	0.93822	0.93943	0.94062	0.94179	0.94295	0.94408
1.6	0.94520	0.94630	0.94738	0.94845	0.94950	0.95053	0.95154	0.95254	0.95352	0.95449
1.7	0.95543	0.95637	0.95728	0.95818	0.95907	0.95994	0.96080	0.96164	0.96246	0.96327
1.8	0.96407	0.96485	0.96562	0.96638	0.96712	0.96784	0.96856	0.96926	0.96995	0.97062
1.9	0.97128	0.97193	0.97257	0.97320	0.97381	0.97441	0.97500	0.97558	0.97615	0.97670
2.0	0.97725	0.97778	0.97831	0.97882	0.97932	0.97982	0.98030	0.98077	0.98124	0.98169
2.1	0.98214	0.98257	0.98300	0.98341	0.98382	0.98422	0.98461	0.98500	0.98537	0.98574
2.2	0.98610	0.98645	0.98679	0.98713	0.98745	0.98778	0.98809	0.98840	0.98870	0.98899
2.3	0.98928	0.98956	0.98983	0.99010	0.99036	0.99061	0.99086	0.99111	0.99134	0.99158
2.4	0.99180	0.99202	0.99224	0.99245	0.99266	0.99286	0.99305	0.99324	0.99343	0.99361
2.5	0.99379	0.99396	0.99413	0.99430	0.99446	0.99461	0.99477	0.99492	0.99506	0.99520
2.6	0.99534	0.99547	0.99560	0.99573	0.99585	0.99598	0.99609	0.99621	0.99632	0.99643
2.7	0.99653	0.99664	0.99674	0.99683	0.99693	0.99702	0.99711	0.99720	0.99728	0.99736
2.8	0.99744	0.99752	0.99760	0.99767	0.99774	0.99781	0.99788	0.99795	0.99801	0.99807
2.9	0.99813	0.99819	0.99825	0.99831	0.99836	0.99841	0.99846	0.99851	0.99856	0.99861
3.0	0.99865	0.99869	0.99874	0.99878	0.99882	0.99886	0.99889	0.99893	0.99896	0.99900
3.1	0.99903	0.99906	0.99910	0.99913	0.99916	0.99918	0.99921	0.99924	0.99926	0.99929
3.2	0.99931	0.99934	0.99936	0.99938	0.99940	0.99942	0.99944	0.99946	0.99948	0.99950
3.3	0.99952	0.99953	0.99955	0.99957	0.99958	0.99960	0.99961	0.99962	0.99964	0.99965
3.4	0.99966	0.99968	0.99969	0.99970	0.99971	0.99972	0.99973	0.99974	0.99975	0.99976
3.5	0.99977	0.99978	0.99978	0.99979	0.99980	0.99981	0.99981	0.99982	0.99983	0.99983
3.6	0.99984	0.99985	0.99985	0.99986	0.99986	0.99987	0.99987	0.99988	0.99988	0.99989
3.7	0.99989	0.99990	0.99990	0.99990	0.99991	0.99991	0.99992	0.99992	0.99992	0.99992
3.8	0.99993	0.99993	0.99993	0.99994	0.99994	0.99994	0.99994	0.99995	0.99995	0.99995
3.9	0.99995	0.99995	0.99996	0.99996	0.99996	0.99996	0.99996	0.99996	0.99997	0.99997
4.0	0.99997	0.99997	0.99997	0.99997	0.99997	0.99997	0.99998	0.99998	0.99998	0.99998
4.1	0.99998	0.99998	0.99998	0.99998	0.99998	0.99998	0.99998	0.99998	0.99999	0.99999