Statistics Masters Comprehensive Exam

April 9, 2011

Student Name: ____________________________

1. Answer 8 out of 12 problems. Mark the problems you selected in the following table.

Problem	1	2	3	4	5	6	7	8	9	10	11	12
Selected												
Scores												

2. Write your answer right after each problem selected, attach more pages if necessary. **Do not** write your answers on the back.
3. Assemble your work in right order and in the original problem order. (Including the ones that you do not select)
4. You can use the N(0,1) distribution table as attached.

1. Let X and Y have joint pdf

$$f(x,y) = C(x+2y), \quad 0 \leq x \leq 1, \quad 0 < y < 1.$$

(a) Find the constant C.

(b) Find the marginal pdf of X.

(c) Find the pdf of $Z = (2X-1)^2$.

2. Let $\{X_1, \ldots, X_n\}$ be a random sample from the population with density $f(x, \theta) = \theta^{-1} e^{-x/\theta}, x > 0, \theta > 0$.

   (a) Derive the UMP (Uniformly Most Powerful) size α test for testing $H_0 : \theta = 1$ versus $H_1 : \theta > 1$.

   (b) Explain how to compute the power function of your test.

3. Let $X_1, \ldots, X_n$ be a random sample from a population with unknown parameter θ.

   (a) If the population has a Normal$(0, \theta)$ distribution, find the maximum likelihood estimator of θ.

   (b) Is the MLE of θ an unbiased statistic for θ? Justify your answer.

   (c) If the population has a Uniform $(0, \theta)$ distribution, find the maximum likelihood estimator of θ.

   (d) Is the MLE of θ an unbiased statistic for θ? Justify your answer.

4. Let $X_1, \ldots, X_n$ is a random sample from Normal(μ_1, σ_1^2) population; and $Y_1, \ldots, Y_m$ is a random sample from Normal(μ_2, σ_2^2) population. Write the best tests (no formal derivation is necessary) of the following hypotheses:

   (a) $H_0 : \mu_1 = \mu_2$ versus $H_1 : \mu_1 \neq \mu_2$. (assume $\sigma_1 = \sigma_2 = \sigma$, and σ is unknown.)

   (b) $H_0 : \sigma_1 = \sigma_2$ versus $H_1 : \sigma_1 \neq \sigma_2$.

   (c) $H_0 : \sigma_1 = 2\sigma_2$ versus $H_1 : \sigma_1 \neq 2\sigma_2$.

5. Let X_i and X_2 be two independent random variables having a chi-squared distribution. with degrees of freedom n_1 and n_2, respectively. Let

$$Y_1 = \frac{X_1}{X_1 + X_2} \quad \text{and} \quad Y_2 = X_1 + X_2.$$

(a) Find the joint p.d.f. of Y_1 and Y_2.

(b) Find the marginal p.d.f. of each of Y_1 and Y_2.

(c) Are Y_1 and Y_2 independent ? Justify your answer.

6. Let X be a random variable with a Beta$(2, 1)$. distribution. Let $Y = \log(X/(1 - X))$.

   (a) Find the distribution function and density function of Y.

   (b) Find $E(e^{-Y})$.

7. Let $X_1, \ldots, X_4$ be a random sample an Normal$(0, \sigma^2)$ population. We wish to test $H_0 : \sigma^2 = 1$ versus $H_1 : \sigma^2 = 4$.

   (a) Consider the following tests procedures

      i. Reject H_0 if $\sum_{i=1}^{4} X_i^2 > C_1$.

      ii. Reject H_0 if $\min\{X_1^2, \ldots, X_4^2\} > C_2$.

   (b) Find C_1 and C_2 so that the probability of Type I error is .05

   (c) Explain how to calculate the power of the tests and compare the two tests.

8. Let $X_1, \ldots, X_n$ be a random sample from a population with density

$$f(x|\theta) = \begin{cases} 2x\theta^2 & , \quad \text{if} \quad 0 < x < \frac{1}{\theta} \\ 0 & , \quad \text{if otherwise} \end{cases}$$

(a) Find the maximum likelihood estimator (MLE) of θ^4.

(b) Find a sufficient statistic for θ.

(c) Is the above MLE a minimal sufficient statistic? Explain fully.

9. Let $X_1, \ldots, X_n$ be a random sample from a Bernoulli (θ), and suppose we put a Beta(α, β) prior distribution on θ. Find Bayes estimators of θ using loss functions

   (a) $L(\theta, a) = (\theta - a)^2$

   (b) $L(\theta, a) = \frac{(\theta-a)^2}{\theta^2(1-\theta)}$

   (c) Is there any relationship between the MLE of θ and above the Bayes estimator of θ? Explain fully.

10. Let X be $N(\mu, \sigma^2)$ and Y be $N(\nu, \sigma^2)$. Suppose that X and Y are independent. Define $U = X + Y$ and $V = X - Y$.

   (a) Show that U and V are independent normal random variables.

   (b) Find the marginal distributions of U and V.

11. Let $X_1, \ldots, X_n$ be a random sample from a distribution with pdf

$$f(x, \theta) = \theta x^{\theta - 1}, \quad 0 < x < 1.$$

(a) Find the maximum likelihood estimator of $\mu = \frac{\theta}{1+\theta}$.

(b) Find a sufficient statistic for θ.

(c) Find the UMVUE of $\mu = \frac{\theta}{1+\theta}$.

12. Let $X_1, \ldots, X_n$ be a random sample from the normal distribution $N(\mu, \sigma^2)$.

(a) To test $H_0 : \mu = 59$ against $H_1 : \mu \neq 59$, where σ^2 is unknown, what is the critical region of size $\alpha = 0.05$ specified by the likelihood ratio test criterion?

(b) If a sample of size $n = 100$ yielded $\bar{X} = 56.13$, and $s^2 = 225$, is H_0 accepted?

Table of P(Z < z), Z ~N(0,1)

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.50000	0.50399	0.50798	0.51197	0.51595	0.51994	0.52392	0.52790	0.53188	0.53586
0.1	0.53983	0.54380	0.54776	0.55172	0.55567	0.55962	0.56356	0.56749	0.57142	0.57535
0.2	0.57926	0.58317	0.58706	0.59095	0.59483	0.59871	0.60257	0.60642	0.61026	0.61409
0.3	0.61791	0.62172	0.62552	0.62930	0.63307	0.63683	0.64058	0.64431	0.64803	0.65173
0.4	0.65542	0.65910	0.66276	0.66640	0.67003	0.67364	0.67724	0.68082	0.68439	0.68793
0.5	0.69146	0.69497	0.69847	0.70194	0.70540	0.70884	0.71226	0.71566	0.71904	0.72240
0.6	0.72575	0.72907	0.73237	0.73565	0.73891	0.74215	0.74537	0.74857	0.75175	0.75490
0.7	0.75804	0.76115	0.76424	0.76730	0.77035	0.77337	0.77637	0.77935	0.78230	0.78524
0.8	0.78814	0.79103	0.79389	0.79673	0.79955	0.80234	0.80511	0.80785	0.81057	0.81327
0.9	0.81594	0.81859	0.82121	0.82381	0.82639	0.82894	0.83147	0.83398	0.83646	0.83891
1.0	0.84134	0.84375	0.84614	0.84849	0.85083	0.85314	0.85543	0.85769	0.85993	0.86214
1.1	0.86433	0.86650	0.86864	0.87076	0.87286	0.87493	0.87698	0.87900	0.88100	0.88298
1.2	0.88493	0.88686	0.88877	0.89065	0.89251	0.89435	0.89617	0.89796	0.89973	0.90147
1.3	0.90320	0.90490	0.90658	0.90824	0.90988	0.91149	0.91309	0.91466	0.91621	0.91774
1.4	0.91924	0.92073	0.92220	0.92364	0.92507	0.92647	0.92785	0.92922	0.93056	0.93189
1.5	0.93319	0.93448	0.93574	0.93699	0.93822	0.93943	0.94062	0.94179	0.94295	0.94408
1.6	0.94520	0.94630	0.94738	0.94845	0.94950	0.95053	0.95154	0.95254	0.95352	0.95449
1.7	0.95543	0.95637	0.95728	0.95818	0.95907	0.95994	0.96080	0.96164	0.96246	0.96327
1.8	0.96407	0.96485	0.96562	0.96638	0.96712	0.96784	0.96856	0.96926	0.96995	0.97062
1.9	0.97128	0.97193	0.97257	0.97320	0.97381	0.97441	0.97500	0.97558	0.97615	0.97670
2.0	0.97725	0.97778	0.97831	0.97882	0.97932	0.97982	0.98030	0.98077	0.98124	0.98169
2.1	0.98214	0.98257	0.98300	0.98341	0.98382	0.98422	0.98461	0.98500	0.98537	0.98574
2.2	0.98610	0.98645	0.98679	0.98713	0.98745	0.98778	0.98809	0.98840	0.98870	0.98899
2.3	0.98928	0.98956	0.98983	0.99010	0.99036	0.99061	0.99086	0.99111	0.99134	0.99158
2.4	0.99180	0.99202	0.99224	0.99245	0.99266	0.99286	0.99305	0.99324	0.99343	0.99361
2.5	0.99379	0.99396	0.99413	0.99430	0.99446	0.99461	0.99477	0.99492	0.99506	0.99520
2.6	0.99534	0.99547	0.99560	0.99573	0.99585	0.99598	0.99609	0.99621	0.99632	0.99643
2.7	0.99653	0.99664	0.99674	0.99683	0.99693	0.99702	0.99711	0.99720	0.99728	0.99736
2.8	0.99744	0.99752	0.99760	0.99767	0.99774	0.99781	0.99788	0.99795	0.99801	0.99807
2.9	0.99813	0.99819	0.99825	0.99831	0.99836	0.99841	0.99846	0.99851	0.99856	0.99861
3.0	0.99865	0.99869	0.99874	0.99878	0.99882	0.99886	0.99889	0.99893	0.99896	0.99900
3.1	0.99903	0.99906	0.99910	0.99913	0.99916	0.99918	0.99921	0.99924	0.99926	0.99929
3.2	0.99931	0.99934	0.99936	0.99938	0.99940	0.99942	0.99944	0.99946	0.99948	0.99950
3.3	0.99952	0.99953	0.99955	0.99957	0.99958	0.99960	0.99961	0.99962	0.99964	0.99965
3.4	0.99966	0.99968	0.99969	0.99970	0.99971	0.99972	0.99973	0.99974	0.99975	0.99976
3.5	0.99977	0.99978	0.99978	0.99979	0.99980	0.99981	0.99981	0.99982	0.99983	0.99983
3.6	0.99984	0.99985	0.99985	0.99986	0.99986	0.99987	0.99987	0.99988	0.99988	0.99989
3.7	0.99989	0.99990	0.99990	0.99990	0.99991	0.99991	0.99992	0.99992	0.99992	0.99992
3.8	0.99993	0.99993	0.99993	0.99994	0.99994	0.99994	0.99994	0.99995	0.99995	0.99995
3.9	0.99995	0.99995	0.99996	0.99996	0.99996	0.99996	0.99996	0.99996	0.99997	0.99997
4.0	0.99997	0.99997	0.99997	0.99997	0.99997	0.99997	0.99998	0.99998	0.99998	0.99998
4.1	0.99998	0.99998	0.99998	0.99998	0.99998	0.99998	0.99998	0.99998	0.99999	0.99999