Statistics Masters Comprehensive Exam

November 5, 2011

Student Name: ______________________________

1. Answer 8 out of 12 problems. Mark the problems you selected in the following table.

Problem	1	2	3	4	5	6	7	8	9	10	11	12
Selected												
Scores												

2. Write your answer right after each problem selected, attach more pages if necessary. **Do not** write your answers on the back.

3. Assemble your work in right order and in the original problem order. (Including the ones that you do not select)

4. You can use the N(0,1) distribution table as attached.

1. Suppose that a person plays a game in which he draws a ball from a box of 10 balls numbered 0 through 9. He then puts the ball back and continue to draw a ball (with replacement) until he draws another number which is equal or higher than the first draw. Let X and Y denote the number drawn in the first and last try, respectively.

   (a) Find the conditional probability distribution of Y given $X = x$, for $x = 0, 1, \cdots, 9$.

   (b) Find the joint probability distribution of X and Y, $P(X = x, Y = y)$.

   (c) Find the probability distribution of Y, $P(Y = y)$.

2. Let X be a random variable with a $U(0,1)$ distribution. Let $Y = \ln(X/(1-X))$.

   (a) Find the cumulative distribution function of Y.

   (b) Find the probability density function of Y.

3. Let $X_1, \dots, X_n$ be iid $N(\mu, \sigma^2)$. For $k < n$, let

$$\bar{X}_k = \frac{1}{k} \sum_{i=1}^{k} X_i$$

$$\bar{X}_{n-k} = \frac{1}{n-k} \sum_{i=k+1}^{n} X_i$$

$$S_k^2 = \frac{1}{k-1} \sum_{i=1}^{k} (X_i - \bar{X}_k)^2$$

$$S_{n-k}^2 = \frac{1}{n-k-1} \sum_{i=k+1}^{n} (X_i - \bar{X}_{n-k})^2$$

(a) What is the distribution of $\sigma^{-2}\left((k-1)S_k^2 + (n-k-1)S_{n-k}^2\right)$?

(b) What is the distribution of $\frac{1}{2}(\bar{X}_k + \bar{X}_{n-k})$?

(c) What is the distribution of $\sigma^{-2}(X_i - \mu)^2$?

(d) What is the distribution of $\frac{S_k^2}{S_{n-k}^2}$?

4. Let $X_1, \ldots, X_{25}$ be a random sample from a normal population with mean $\mu = 0$ and variance $\sigma^2 = 16$. Let $\bar{X} = \frac{X_1 + \ldots + X_{25}}{25}$ and let $S^2 = \sum_{i=1}^{25}(X_i - \bar{X})^2/24$

   (a) Find $E(\bar{X}^2 S^2)$

   (b) Find a so that $\frac{a\bar{X}^2}{S^2}$ has an F distribution. What are the degrees of freedom?

5. Let $X_1, \ldots, X_n$ be a random sample from a population with unknown parameter θ.

   (a) If the population has a Normal$(0, \theta)$ distribution, find the maximum likelihood estimator of θ.

   (b) Is the MLE of θ a sufficient statistic for θ?

   (c) If the population has a Uniform $(0, \theta)$ distribution, find the maximum likelihood estimator of θ.

   (d) Is the MLE of θ a sufficient statistic for θ?

6. Let $X_1, \ldots, X_{16}$ be a random sample an Normal$(\theta, 1)$ population. We wish to test $H_0 : \theta = 1$ versus $H_1 : \theta = 2$.

   (a) Suppose we reject H_0 if $\sum_{i=1}^{16} X_i > C$. Find C so that the probability of Type I error is .05

   (b) Calculate the power of the test.

7. Suppose that the random sample $X_1, ..., X_n$ taken from Poisson distribution with an unknown mean θ.

   (a) Find the MLE of $e^{-\theta}$.

   (b) Find the UMVUE of $e^{-\theta}$.

   (c) Compute the Cramer-Rao lower bound of unbiased estimators of $e^{-\theta}$.

8. Let $X_1, X_2, X_3 \cdots, X_n$ be a random sample from a population with density

$$f(x;\theta) = \theta x^{\theta-1}, 0 < x < 1.$$

Construct a uniformly most powerful test of

$$H_0 : \theta = 6 \text{ against } H_1 : \theta < 6.$$

9. Let $X_1, \ldots, X_n$ be a random sample an Exponential population with parameter θ. That is,

$$f(x|\theta) = \theta e^{-\theta x}, \quad x > 0$$

Suppose we put a Gamma (α, β) prior on θ.

(a) Show that this prior is conjugate.

(b) Find the Bayes estimator of θ if we use loss function $L(\theta, a) = (\theta - a)^2$.

10. Assume that in the city of Memphis, 5% of all households have 4PC's, 40% of all households have 3PC's, 30% have 2PC's, 20% have 1PC, and 5% have no PC's. In a survey of the households in the city of Memphis, a sample of 100 households is randomly selected. Assuming that the hypothesis is true, what is the probability that the **average number** of PC's among the selected households is less than 2?

11. Let X be a continuous variable with distribution function F.

    (a) Prove that $F(X)$ has a uniform distribution on $(0, 1)$.

    (b) Find the distribution of $2 \min\{F(X), 1 - F(X)\}$.

12. Let $\{X_1, \ldots, X_n\}$ be a random sample from the population with density $f(x;\theta) = \theta^x(1-\theta)^{1-x}, x = 0, 1; 0 < \theta < 1$.

   (a) Put $\bar{X} = \frac{Y}{n}$, where $Y = \sum_{i=1}^{n} X_i$. Show that $\bar{X}$ is the UMUVE (Uniformly Minimum Variance Unbiased Estimator) of θ.

   (b) Find the UMUVE of θ^2.

Table of P(Z < z), Z ~N(0,1)

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.50000	0.50399	0.50798	0.51197	0.51595	0.51994	0.52392	0.52790	0.53188	0.53586
0.1	0.53983	0.54380	0.54776	0.55172	0.55567	0.55962	0.56356	0.56749	0.57142	0.57535
0.2	0.57926	0.58317	0.58706	0.59095	0.59483	0.59871	0.60257	0.60642	0.61026	0.61409
0.3	0.61791	0.62172	0.62552	0.62930	0.63307	0.63683	0.64058	0.64431	0.64803	0.65173
0.4	0.65542	0.65910	0.66276	0.66640	0.67003	0.67364	0.67724	0.68082	0.68439	0.68793
0.5	0.69146	0.69497	0.69847	0.70194	0.70540	0.70884	0.71226	0.71566	0.71904	0.72240
0.6	0.72575	0.72907	0.73237	0.73565	0.73891	0.74215	0.74537	0.74857	0.75175	0.75490
0.7	0.75804	0.76115	0.76424	0.76730	0.77035	0.77337	0.77637	0.77935	0.78230	0.78524
0.8	0.78814	0.79103	0.79389	0.79673	0.79955	0.80234	0.80511	0.80785	0.81057	0.81327
0.9	0.81594	0.81859	0.82121	0.82381	0.82639	0.82894	0.83147	0.83398	0.83646	0.83891
1.0	0.84134	0.84375	0.84614	0.84849	0.85083	0.85314	0.85543	0.85769	0.85993	0.86214
1.1	0.86433	0.86650	0.86864	0.87076	0.87286	0.87493	0.87698	0.87900	0.88100	0.88298
1.2	0.88493	0.88686	0.88877	0.89065	0.89251	0.89435	0.89617	0.89796	0.89973	0.90147
1.3	0.90320	0.90490	0.90658	0.90824	0.90988	0.91149	0.91309	0.91466	0.91621	0.91774
1.4	0.91924	0.92073	0.92220	0.92364	0.92507	0.92647	0.92785	0.92922	0.93056	0.93189
1.5	0.93319	0.93448	0.93574	0.93699	0.93822	0.93943	0.94062	0.94179	0.94295	0.94408
1.6	0.94520	0.94630	0.94738	0.94845	0.94950	0.95053	0.95154	0.95254	0.95352	0.95449
1.7	0.95543	0.95637	0.95728	0.95818	0.95907	0.95994	0.96080	0.96164	0.96246	0.96327
1.8	0.96407	0.96485	0.96562	0.96638	0.96712	0.96784	0.96856	0.96926	0.96995	0.97062
1.9	0.97128	0.97193	0.97257	0.97320	0.97381	0.97441	0.97500	0.97558	0.97615	0.97670
2.0	0.97725	0.97778	0.97831	0.97882	0.97932	0.97982	0.98030	0.98077	0.98124	0.98169
2.1	0.98214	0.98257	0.98300	0.98341	0.98382	0.98422	0.98461	0.98500	0.98537	0.98574
2.2	0.98610	0.98645	0.98679	0.98713	0.98745	0.98778	0.98809	0.98840	0.98870	0.98899
2.3	0.98928	0.98956	0.98983	0.99010	0.99036	0.99061	0.99086	0.99111	0.99134	0.99158
2.4	0.99180	0.99202	0.99224	0.99245	0.99266	0.99286	0.99305	0.99324	0.99343	0.99361
2.5	0.99379	0.99396	0.99413	0.99430	0.99446	0.99461	0.99477	0.99492	0.99506	0.99520
2.6	0.99534	0.99547	0.99560	0.99573	0.99585	0.99598	0.99609	0.99621	0.99632	0.99643
2.7	0.99653	0.99664	0.99674	0.99683	0.99693	0.99702	0.99711	0.99720	0.99728	0.99736
2.8	0.99744	0.99752	0.99760	0.99767	0.99774	0.99781	0.99788	0.99795	0.99801	0.99807
2.9	0.99813	0.99819	0.99825	0.99831	0.99836	0.99841	0.99846	0.99851	0.99856	0.99861
3.0	0.99865	0.99869	0.99874	0.99878	0.99882	0.99886	0.99889	0.99893	0.99896	0.99900
3.1	0.99903	0.99906	0.99910	0.99913	0.99916	0.99918	0.99921	0.99924	0.99926	0.99929
3.2	0.99931	0.99934	0.99936	0.99938	0.99940	0.99942	0.99944	0.99946	0.99948	0.99950
3.3	0.99952	0.99953	0.99955	0.99957	0.99958	0.99960	0.99961	0.99962	0.99964	0.99965
3.4	0.99966	0.99968	0.99969	0.99970	0.99971	0.99972	0.99973	0.99974	0.99975	0.99976
3.5	0.99977	0.99978	0.99978	0.99979	0.99980	0.99981	0.99981	0.99982	0.99983	0.99983
3.6	0.99984	0.99985	0.99985	0.99986	0.99986	0.99987	0.99987	0.99988	0.99988	0.99989
3.7	0.99989	0.99990	0.99990	0.99990	0.99991	0.99991	0.99992	0.99992	0.99992	0.99992
3.8	0.99993	0.99993	0.99993	0.99994	0.99994	0.99994	0.99994	0.99995	0.99995	0.99995
3.9	0.99995	0.99995	0.99996	0.99996	0.99996	0.99996	0.99996	0.99996	0.99997	0.99997
4.0	0.99997	0.99997	0.99997	0.99997	0.99997	0.99997	0.99998	0.99998	0.99998	0.99998
4.1	0.99998	0.99998	0.99998	0.99998	0.99998	0.99998	0.99998	0.99998	0.99999	0.99999