Statistics Masters Comprehensive Exam

November 13, 2010

Student Name: ______________________________

1. Answer 8 out of 12 problems. Mark the problems you selected in the following table.

Problem	1	2	3	4	5	6	7	8	9	10	11	12
Selected												
Scores												

2. Write your answer right after each problem selected, attach more pages if necessary. **Do not** write your answers on the back.

3. Assemble your work in right order and in the original problem order. (Including the ones that you do not select)

4. You can use the N(0,1) distribution table as attached.

1. Suppose that a person plays a game in which he draws a ball from a box of 6 balls numbered 1 through 6. He then puts the ball back and continue to draw a ball (with replacement) until he draws another number which is equal or higher than the first draw. Let X and Y denote the number drawn in the first and last try, respectively.

   (a) Find the probability distribution of X(the first draw).

   (b) Find the probability distribution of Y(the final draw).

   (c) Find the probability of terminating the game at the third draw.

2. Let $X_1, X_2, \cdots, X_{100}$ be i.i.d. $\sim N(\theta, \sigma^2)$, where σ is known. A 95% C.I. for θ is given as $\bar{X} \pm 1.96\sigma/\sqrt{100}$. What percentage of the observations $X_1, X_2, ..X_{100}$ would you expect to fall in this interval ? Justify your answer.

3. Let X and Y be two random variables with joint density

$$f(x,y) = \begin{cases} e^{-(x+y)}, & 0 < x < \infty, 0 < y < \infty \\ 0, & \text{otherwise} \end{cases}$$

Define $U = X + Y, V = X/(X + Y)$.

(a) Find the the joint density of U and V.

(b) Find the the marginal density of U.

(c) Find the the marginal density of V.

4. Suppose that the random sample $X_1, ..., X_n$ taken from Poisson distribution with an unknown mean θ.

   (a) Find the MLE of $e^{-\theta}$.

   (b) Find the UMVUE of $e^{-\theta}$.

   (c) Compute the Cramer-Rao lower bound of unbiased estimators of $e^{-\theta}$.

5. Let $X_1, X_2 \ldots, X_n$ be a sequence of independent normal random variables with $N(\mu, \sigma^2)$ distribution. Derive the likelihood ratio test for testing $H_0 : \mu = \mu_0$ vs. $H_1 : \mu \neq \mu_0$.

6. Let $X_1, X_2, X_3 \cdots, X_{10}$ be a random sample from a population with density

$$f(x|\theta) = \theta e^{-\theta x}, x > 0.$$

Construct a uniformly most powerful size 0.05 test of

$$H_0 : \theta = 1 \text{ against } H_1 : \theta > 1.$$

7. Let $X_1, X_2 \ldots,$ be a sequence of independent exponential random variables with density f given by
$$f(x) = \lambda e^{-\lambda x},$$
Let N be a geometric random variable with probability function
$$P(N = n) = p(1-p)^{n-1},$$
where $n = 1, 2, \ldots$ Assume that the $X's$ and N are independent. If $X_{1,N} = \min(X_1, \ldots, X_N)$, find

(a) $P(X_{1,N} \leq a | N = n)$.

(b) $P(X_{1,N} > a)$.

8. Let $X_1, \ldots, X_n$ be a independent random variables such that $X_i \sim Normal(\theta, \sigma^2/a_i)$, where $\sum_{i=1}^n a_i = 1$.

   (a) Find the maximum likelihood estimators of θ and σ^2.

   (b) Is the MLE of θ unbiased?

9. Let $X_1, \ldots, X_n$ be a random sample an Exponential population with parameter θ. That is,

$$f(x|\theta) = \theta e^{-\theta x}, \quad x > 0$$

Suppose we put a Gamma (α, β) prior on θ.

(a) Show that this prior is conjugate.

(b) Find the Bayes estimator of θ if we use loss function $L(\theta, a) = (\theta - a)^2$.

10. Let $\{X_1, \ldots, X_4\}$ be independently and identically distributed normal variables with mean μ and variance σ^2. Put $Y_1 = X_1 + X_2 + X_3$ and $Y_2 = X_2 + X_3 + X_4$.

   (a) Obtain the joint probability density function (pdf) of (Y_1, Y_2).

   (b) What is the pdf of $Z = (Y_1 - Y_2)^2$?

11. Let $\{X_1, \ldots, X_n\}$ be a random sample from the population with probablity density function given by $f(x; \theta_1, \theta_2) = \frac{1}{\theta_2 - \theta_1}$ if $\theta_1 < x < \theta_2$ and $= 0$ for otherwise.

   (a) Let $X_{(1)}$=Minimum of $(X_1, \ldots, X_n)$ and $X_{(n)}$=Maximum of $(X_1, \ldots, X_n)$. Show that $(X_{(1)}, X_{(n)})$ is a set of sufficient and complete statistics for $(\theta_i, i = 1, 2)$.

   (b) Derive the UMVUE (Uniformly Minimum Varianced and Unbiased Estimator) of $\phi = \theta_2 - \theta_1$.

12. Let $\{X_1, \ldots, X_{16}\}$ be a random sample from the normal distribution with mean θ and variance 4. Consider the null hypothesis $H_0 : \theta = 0$ versus the alternative hypothesis $H_1 : \theta = 2$.

    (a) Derive the size-0.025 MP (Most Powerful) test for testing H_0 vs H_1.

    (b) Assume that the observed sample mean is 2. Based on this observed data, obtain the p-value of your test. From this analysis, what conclusion you will make on H_0?

    (c) Derive the power of your test.

Table of $P(Z < z)$, $Z \sim N(0,1)$

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.50000	0.50399	0.50798	0.51197	0.51595	0.51994	0.52392	0.52790	0.53188	0.53586
0.1	0.53983	0.54380	0.54776	0.55172	0.55567	0.55962	0.56356	0.56749	0.57142	0.57535
0.2	0.57926	0.58317	0.58706	0.59095	0.59483	0.59871	0.60257	0.60642	0.61026	0.61409
0.3	0.61791	0.62172	0.62552	0.62930	0.63307	0.63683	0.64058	0.64431	0.64803	0.65173
0.4	0.65542	0.65910	0.66276	0.66640	0.67003	0.67364	0.67724	0.68082	0.68439	0.68793
0.5	0.69146	0.69497	0.69847	0.70194	0.70540	0.70884	0.71226	0.71566	0.71904	0.72240
0.6	0.72575	0.72907	0.73237	0.73565	0.73891	0.74215	0.74537	0.74857	0.75175	0.75490
0.7	0.75804	0.76115	0.76424	0.76730	0.77035	0.77337	0.77637	0.77935	0.78230	0.78524
0.8	0.78814	0.79103	0.79389	0.79673	0.79955	0.80234	0.80511	0.80785	0.81057	0.81327
0.9	0.81594	0.81859	0.82121	0.82381	0.82639	0.82894	0.83147	0.83398	0.83646	0.83891
1.0	0.84134	0.84375	0.84614	0.84849	0.85083	0.85314	0.85543	0.85769	0.85993	0.86214
1.1	0.86433	0.86650	0.86864	0.87076	0.87286	0.87493	0.87698	0.87900	0.88100	0.88298
1.2	0.88493	0.88686	0.88877	0.89065	0.89251	0.89435	0.89617	0.89796	0.89973	0.90147
1.3	0.90320	0.90490	0.90658	0.90824	0.90988	0.91149	0.91309	0.91466	0.91621	0.91774
1.4	0.91924	0.92073	0.92220	0.92364	0.92507	0.92647	0.92785	0.92922	0.93056	0.93189
1.5	0.93319	0.93448	0.93574	0.93699	0.93822	0.93943	0.94062	0.94179	0.94295	0.94408
1.6	0.94520	0.94630	0.94738	0.94845	0.94950	0.95053	0.95154	0.95254	0.95352	0.95449
1.7	0.95543	0.95637	0.95728	0.95818	0.95907	0.95994	0.96080	0.96164	0.96246	0.96327
1.8	0.96407	0.96485	0.96562	0.96638	0.96712	0.96784	0.96856	0.96926	0.96995	0.97062
1.9	0.97128	0.97193	0.97257	0.97320	0.97381	0.97441	0.97500	0.97558	0.97615	0.97670
2.0	0.97725	0.97778	0.97831	0.97882	0.97932	0.97982	0.98030	0.98077	0.98124	0.98169
2.1	0.98214	0.98257	0.98300	0.98341	0.98382	0.98422	0.98461	0.98500	0.98537	0.98574
2.2	0.98610	0.98645	0.98679	0.98713	0.98745	0.98778	0.98809	0.98840	0.98870	0.98899
2.3	0.98928	0.98956	0.98983	0.99010	0.99036	0.99061	0.99086	0.99111	0.99134	0.99158
2.4	0.99180	0.99202	0.99224	0.99245	0.99266	0.99286	0.99305	0.99324	0.99343	0.99361
2.5	0.99379	0.99396	0.99413	0.99430	0.99446	0.99461	0.99477	0.99492	0.99506	0.99520
2.6	0.99534	0.99547	0.99560	0.99573	0.99585	0.99598	0.99609	0.99621	0.99632	0.99643
2.7	0.99653	0.99664	0.99674	0.99683	0.99693	0.99702	0.99711	0.99720	0.99728	0.99736
2.8	0.99744	0.99752	0.99760	0.99767	0.99774	0.99781	0.99788	0.99795	0.99801	0.99807
2.9	0.99813	0.99819	0.99825	0.99831	0.99836	0.99841	0.99846	0.99851	0.99856	0.99861
3.0	0.99865	0.99869	0.99874	0.99878	0.99882	0.99886	0.99889	0.99893	0.99896	0.99900
3.1	0.99903	0.99906	0.99910	0.99913	0.99916	0.99918	0.99921	0.99924	0.99926	0.99929
3.2	0.99931	0.99934	0.99936	0.99938	0.99940	0.99942	0.99944	0.99946	0.99948	0.99950
3.3	0.99952	0.99953	0.99955	0.99957	0.99958	0.99960	0.99961	0.99962	0.99964	0.99965
3.4	0.99966	0.99968	0.99969	0.99970	0.99971	0.99972	0.99973	0.99974	0.99975	0.99976
3.5	0.99977	0.99978	0.99978	0.99979	0.99980	0.99981	0.99981	0.99982	0.99983	0.99983
3.6	0.99984	0.99985	0.99985	0.99986	0.99986	0.99987	0.99987	0.99988	0.99988	0.99989
3.7	0.99989	0.99990	0.99990	0.99990	0.99991	0.99991	0.99992	0.99992	0.99992	0.99992
3.8	0.99993	0.99993	0.99993	0.99994	0.99994	0.99994	0.99994	0.99995	0.99995	0.99995
3.9	0.99995	0.99995	0.99996	0.99996	0.99996	0.99996	0.99996	0.99996	0.99997	0.99997
4.0	0.99997	0.99997	0.99997	0.99997	0.99997	0.99997	0.99998	0.99998	0.99998	0.99998
4.1	0.99998	0.99998	0.99998	0.99998	0.99998	0.99998	0.99998	0.99998	0.99999	0.99999