Statistics Masters Comprehensive Exam

November 7, 2009

Student Name: ______________________________

1. Answer 8 out of 12 problems. Mark the problems you selected in the following table.

Problem	1	2	3	4	5	6	7	8	9	10	11	12
Selected												
Scores												

2. Write your answer right after each problem selected, attach more pages if necessary. **Do not** write your answers on the back.

3. Assemble your work in right order and in the original problem order. (Including the ones that you do not select)

1. Let X and Y be two random variables with joint density

$$f(x,y) = \begin{cases} 1, & 0 < x < 1, 0 < y < 1 \\ 0, & \text{otherwise} \end{cases}$$

Define $U = X + Y, V = X/(X + Y)$.

   (a) Find the the joint density of U and V.

   (b) Find the the marginal density of U.

   (c) Find the the marginal density of V.

2. Let $(X_1, X_2, X_3) \sim \text{Trinomial}(n, p_1, p_2, p_3)$, that is,

$$P(X_1 = x_1, X_2 = x_2, X_3 = x_3) = \frac{n!}{x_1! x_2! x_3!} p_1^{x_1} p_2^{x_2} p_3^{x_3},$$

where $p_3 = 1 - p_1 - p_2,\ x_3 = n - x_1 - x_2$, find $P(X_1 = k | X_3 = m)$.

3. Let $X_1, \ldots, X_4$ be a random sample an Normal$(\theta, 1)$ population. We wish to test $H_0 : \theta = 2$ versus $H_1 : \theta = 1$.

   (a) Consider the following tests procedures

      i. Reject H_0 if $\sum_{i=1}^4 X_i < C_1$.

      ii. Reject H_0 if $\min\{X_1, \ldots, X_4\} < C_2$.

   (b) Find C_1 and C_2 so that the probability of Type I error is .05

   (c) Calculate the power of the tests and compare the two tests.

4. Let $X_1, \ldots, X_n$ be a random sample from a population with density

$$f(x|\theta) = \begin{cases} \left(\frac{\theta}{x}\right)^{\theta+1} & , \quad \text{if} \quad x > \theta \\ 0 & , \quad \text{if otherwise,} \end{cases}$$

where $\theta > 4$.

(a) Find the maximum likelihood estimator (MLE) of θ^4.

(b) Is the above MLE a minimal sufficient statistics? Explain fully.

5. Let X_1, X_2 be independent random variables.

   (a) If $X_1 \sim Poisson(5)$ and $X_2 \sim Poisson(2)$, find $P(X_1 = 1 | X_1 + X_2 = 2)$.

   (b) If $X|\theta \sim Poisson(\theta)$ and $\theta \sim Gamma(\alpha, \beta)$, find $E(\theta^4 | X = 2)$.

6. Let $X_1, \ldots, X_n$ be a random sample from a Bernoulli (θ), and suppose we put a Beta(α, β) prior distribution on θ.

   Find Bayes estimators of θ using loss functions

   (a) $L(\theta, a) = (\theta - a)^2$

   (b) $L(\theta, a) = \frac{(\theta - a)^2}{\theta(1-\theta)}$,

   (c) Show that with the second loss function, the MLE of θ is the Bayes estimator of θ, when $\alpha = \beta = 1$.

7. The probability integral $I_d(a,b)$ for $(a>0, b>0)$ and for $1>d>0$ is defined as the incomplete Beta-integral if $I_d(a,b)$ is given by

$$I_d(a,b) = \frac{1}{B(a,b)} \int_0^d x^{a-1}(1-x)^{b-1}dx$$

where $B(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}dx$. Express the probability $P(X>c)$ $(c>0)$ in terms of incomplete Beta-integral in each of the following.

(a) $X \sim F(f_1, f_2)$, a central F-distribution with degrees of freedom (f_1, f_2), where $f_i > 0$ are integers.

(b) $X \sim t_f$, a central t-distribution with degrees of freedom f, where $f>0$ is an integer.

8. Let $\{X_1, \ldots, X_n\}$ be a random sample from the Bernoulli distribution with density $f(x, \theta) = \theta^x (1-\theta)^{1-x}, x = 0, 1$, where $0 < \theta < 1$. Let the density of the prior distribution be given by $P(\theta) \propto \theta^{a-1}(1-\theta)^{b-1}, 0 \leq \theta \leq 1$, where $a > 0, b > 0$.

   (a) Given the loss function as $l(\theta, s) = (s - \theta)^2$ for the estimator s of θ, derive the Bayese estimator of θ.

   (b) Derive the maximum likelihood estimator (MLE) $\hat{\theta}$ of θ and compare the MLE with the Bayese estimator.

9. Let $\{X_1, \ldots, X_n\}$ be a random sample from the density $f(x;\theta) = \frac{1}{\theta_2 - \theta_1}, \theta_1 < x < \theta_2$.

   (a) Derive the MLE (Maximum Likelihood Estimator) $\hat{\theta}_1$ and $\hat{\theta}_2$ of θ_1 and θ_2 respectively. Are your estimators forming a set of sufficient and complete statistics for $\{\theta_1, \theta_2\}$?

   (b) Derive the UMVUE (Uniformly Minimum Varianced Unbiased Estimator) of $\theta_2 - \theta_1$.

10. Let $(X_1, Y_1), \ldots, (X_n, Y_n)$ be independently and identically distributed as (X, Y), where (X, Y) follows a bivariate normal distribution with means $EX = \mu_1$ and $EY = \mu_2$ and with the variances $Var(X) = \sigma_1^2$ and $Var(Y) = \sigma_2^2$ respectively. Assume that X is un-correlated with Y.

    (a) Derive the likelihood ratio test (LRT) for testing $H_0 : \sigma_1^2 = \sigma_2^2$ against the alternative hypothesis $H_1 : \sigma_1^2 \neq \sigma_2^2$.

    (b) Derive the probability distribution of your test statistic under H_1. What is the power function of the LRT test?

11. Let $X_1, X_2, \cdots, X_n$ be a random sample from a distribution with pmf

$$f(x) = \theta^x(1-\theta)^{1-x}, x = 0, 1; 0 < \theta < 1.$$

Find UMVUEs for

(a) θ

(b) θ^2

(c) $\theta(1-\theta)$.

12. A sample of size n is drawn from each of k normal populations with the same but unown variance. Describe a procedure to derive the likelihood ratio test for testing the hypothesis that the means are all zero.