Statistics Masters Comprehensive Exam

November 18, 2006

Student Name: ______________________________

1. Answer 8 out of 12 problems. Mark the problems you selected in the following table.

Problem	1	2	3	4	5	6	7	8	9	10	11	12
Selected												
Scores												

2. Write your answer right after each problem selected, attach more pages if necessary.

3. Assemble your work in right order and in the original problem order.

1. Let X_1, X_2 be two independent random variables from $U(0,1)$.

   (a) Find the joint p.d.f. of $Y_1 = X_1 + X_2$ and $Y_2 = X_1 - X_2$.

   (b) Find the marginal p.d.f. of Y_1.

   (c) Find the marginal p.d.f. of Y_2.

2. Let X be a random variable with the p.d.f.

$$f(x;\theta) = \theta\,(1-x)^{\theta-1}, \quad 0 < x < 1, 0 < \theta < \infty.$$

(a) Find the p.d.f. of $Y = -\ln(1-X)$.

(b) Find the p.d.f. of $Y = \ln(X/(1-X))$.

3. Let $X_1, X_2, \cdots, X_n$ be a random sample taken from the distribution with the p.d.f.

$$f(x;\theta) = \theta\ x^{\theta-1}, \quad 0 < x < 1, 0 < \theta < \infty.$$

(a) Derive the moment estimator of θ.

(b) Derive the maximum likelihood estimator of θ.

4. Suppose $X_1, X_2, X_3, ...X_{32}$ be a random sample with a *beta* distribution with the p.d.f. $f(x) = 2x, 0 < x < 1$.

   (a) Let Y be the number of these random variables ($X_i, i = 1, 2, ..., 32$) whose values less than 0.5. Find $P(Y \leq 10)$.

   (b) Approximate $P(\sum_{i=1}^{14} X_i > \sum_{i=15}^{32} X_i)$.

5. Let X be a random variable whose probability mass function under H_0 and H_1 is given by

x	1	2	3	4	5	6	7
$f(x\|H_0)$	0.01	0.01	0.01	0.01	0.01	0.01	0.94
$f(x\|H_1)$	0.06	0.05	0.04	0.03	0.02	0.01	0.79

(a) Use the Neyman-Pearson Lemma to find the most powerful test for H_0 vs. H_1 with size $\alpha = 0.05$.

(b) Compute the probability of Type II error for the above test.

6. Let X_1 and X_2 be independent random variables with $X_i \sim \mathcal{N}(\mu_i, \sigma_i^2), i = 1, 2$. Let $Y = X_1 + X_2$ and $Z = X_1 X_2$

   (a) Find the distribution of X_1 given $Y = y$.

   (b) Find $Cov(Y, Z)$

   (c) If $\mu_1 = \mu_2 = 0$, are Y and Z independent? (Fully explain your answer.)

7. Let $X_{i1}, \ldots, X_{in_i}, i = 1, 2$ be independent random samples from exponential distributions with means $\theta_i, i = 1, 2$.

   (a) Derive a likelihood ratio test for testing $H_0 : \theta_1 = \theta_2$ versus $H_1 : \theta_1 \neq \theta_2$

   (b) Show that the test statistic can be expressed as

$$T = \frac{\sum_{j=1}^{n_1} X_{1j}}{\sum_{j=1}^{n_1} X_{1j} + \sum_{j=1}^{n_2} X_{2j}}$$

   (c) Find the distribution of T when H_0 is true.

8. Let $X_1, \ldots, X_n$ be a random sample from a population with density

$$f(x;\theta) = 3\theta x^2 e^{-\theta x^3}, \quad x > 0.$$

(a) Let M be the median of this distribution. Find the MLE of M.

(b) Suppose we put a $Gamma(\alpha, \beta)$ prior distribution on θ, find the Bayes estimate of θ using squared error loss function.

9. Let $\{X_1, \ldots, X_n\}$ be independently and identically distributed normal variables with mean 0 and variance 1. Put $Y_1 = \frac{1}{n}\sum_{i=1}^{n} X_i$ and $Y_2 = \sum_{i=1}^{n}(X_i - Y_1)^2$.

   (a) Show that Y_1 and Y_2 are independently distributed of each other stochastically.

   (b) What is the sampling distribution of Y_2?

10. Let $\{X_1, \dots, X_n\}$ be a random sample from the density $f(x; \theta) = e^{-\theta}\theta^x/x!, x = 0, 1, \dots, \infty, \theta > 0$.

(a) Obtain a sufficient and complete statistic for θ.

(b) Derive the UMVUE(Uniformly Minimum Varianced and Unbiased estimator) of $P\{X = 1\}$.

11. Let $\{X_1, \ldots, X_n\}$ be a random sample from the density $f(x, \theta) = \theta^{-1}e^{-x/\theta}, 0 < x; f(x, \theta) = 0$, if $x \leq 0.(\theta > 0)$

    (a) Derive the size-α UMP (Uniformly Most Powerful) test for testing $H_0 : \theta = 1$ vs $H_1 : \theta > 1$.

    (b) Assume that the observed sample mean is 0.98 and n=20. Based on this observed data, obtain the p-value of your test. From on this analysis, what conclusions you will make?

12. Let $\{X_1, \ldots, X_{10}\}$ be a random sample from the normal density with mean μ and variance 1. Let the prior distribution of μ be given by $P(\mu) \propto e^{-\frac{1}{8}(\mu-2)^2}$, μ real.

    Assume that the observed value of the sample mean is 1.5. Derive the 95% HPD (Highest Posterior Density) interval of μ.