Statistics Masters Comprehensive Exam

November 1, 2003

Student Name: ______________________________

1. Answer 8 out of 12 problems. Mark the problems you selected in the following table.

1	2	3	4	5	6	7	8	9	10	11	12

2. Write your answer right after each problem selected, attach more pages if necessary.

3. Assemble your work in right order and in the original problem order.

1. Let X_1 and X_2 have independent Gamma distributions with parameters (α, θ) and (β, θ), respectively. That is, the joint p.d.f. of X_1 and X_2 is

$$f(x_1, x_2) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)\theta^{\alpha+\beta}} x_1^{\alpha-1} x_2^{\beta-1} e^{-\frac{x_1+x_2}{\theta}}, \quad 0 < x_1, x_2 < \infty.$$

Consider

$$Y_1 = \frac{X_1}{X_1 + X_2}, \quad Y_2 = X_1 + X_2.$$

(a) Find the joint p.d.f. of Y_1 and Y_2.

(b) Find the marginal p.d.f. of Y_1.

2. Let $X_1, \ldots, X_n$ be a independent random variables such that $X_i \sim Normal(\theta a_i, 1)$, where $a_i, i = 1, \cdots, n$ are some known constants.

   (a) Find the maximum likelihood estimator of θ.

   (b) Use this MLE of θ to construct a $100(1-\alpha)$ confidence interval for θ.

3. Let $W_1 < W_2 < \cdots < W_n$ be the order statistics of n independent observations from a $U(0, 1)$ distribution.

   (a) For $1 \leq r \leq n$, find the p.d.f. of W_r, the r-th order statistics.

   (b) Use the above results to verify that $E(W_1) = \frac{1}{n+1}$ and $E(W_n) = \frac{n}{n+1}$.

4. Let $(X_1, Y_1), \ldots, (X_n, Y_n)$ be independently and identically distributed as (X, Y), where (X, Y) follows a bivariate normal distribution with means $E(X) = \mu_1$ and $E(Y) = \mu_2$ and with the variances $Var(X) = \sigma_1^2$ and $Var(Y) = \sigma_2^2$ respectively. Assume that X is un-correlated with Y.

   (a) Derive the likelihood ratio test (LRT) for testing $H_0 : \sigma_1^2 = \sigma_2^2$ against the alternative hypothesis $H_1 : \sigma_1^2 \neq \sigma_2^2$.

   (b) Derive the probability distribution of your test statistic under H_1. What is the power function of the LRT test?

5. Let $Y_i \sim N(\alpha + \beta x_i, \sigma^2)$ independently for $i = 1, \ldots, n$, where x_i's are the values of a non-random covariate.

   (a) Derive the maximum likelihood estimators of α, β, and σ^2.

   (b) Let $\hat{\alpha}$ and $\hat{\beta}$ be the maximum likelihood estimators for α and β, respectively. Show that

$$\sum_{i=1}^{n} (Y_i - \hat{\alpha} - \hat{\beta} x_i) x_i = 0.$$

6. Let $\{X_1, \ldots, X_n\}$ be a random sample from the Bernoulli distribution with density $f(x, \theta) = \theta^x (1-\theta)^{1-x}, x = 0, 1$, where $0 < \theta < 1$.

   (a) Find the UMVUE of θ^2.

   (b) Let the density of the prior distribution be given by $P(\theta) \propto \theta^{a-1}(1-\theta)^{b-1}, 0 \leq \theta \leq 1$, where $a > 0, b > 0$. Given the loss function as $l(\theta, s) = (s - \theta)^2$ for the estimator s of θ, derive the Bayes estimator of θ.

7. Let X be a continuous variable with distribution function F.

   (a) Prove that $F(X)$ has a uniform distribution on $(0, 1)$.

   (b) Find the distribution of $2\min\{F(X), 1 - F(X)\}$.

8. Let $\{X_1, \ldots, X_n\}$ be a random sample from $N(0, \sigma^2)$.

(a) Show that

$$C = \left\{ (x_1, \cdots, x_n) | \sum_{i=1}^{n} x_i^2 \geq c \right\}$$

is a best critical region for testing $H_0 : \sigma^2 = 4$ against $H_1 : \sigma^2 = 16$.

(b) If $n = 15$. find the value of c so that $\alpha = 0.05$.

9. Let $X_1, \ldots, X_n$ be a random sample an Exponential population with parameter λ. That is,

$$f(x; \lambda) = \lambda e^{-\lambda x}, \quad x > 0$$

We wish to test $H_0 : \lambda = 1$ versus $H_1 : \lambda > 1$. Consider a test which rejects H_0 when $X_{(1)} < C$, where $X_{(1)} = \min(X_1, \ldots, X_n)$.

Find the value of C so that probability of type I error of this procedure is .05.

10. Suppose that a random variable X has the following p.d.f.

$$f(x|\theta) = \begin{cases} \theta x^{\theta-1} & \text{when } 0 < x < 1, \\ 0 & \text{otherwise} \end{cases}$$

where $0 < \theta < \infty$.

(a) Determine the distribution of $Y = -\theta \log X$

(b) If $X_1, \ldots, X_n$ is a random sample from the above p.d.f. construct a $(1-\alpha)100\%$ confidence interval for θ. (The answer should be in terms of tabulated percentiles)

11. Suppose that X_1, X_2 and X_3 are independently distributed of each other. For $i = 1, 2, 3$, let the probability density function X_i be

$$f_i(x) = \begin{cases} \lambda_i e^{-\lambda_i x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

(a) Find $P(X_1 < X_2)$.

(b) Find $P(X_1 < \min(X_2, X_3))$.

(c) Find $P(X_1 < X_2 < X_3)$.

12. Let $X_1, \ldots, X_n$ be a random sample from a population with density

$$f(x;\theta) = \begin{cases} \theta x^{\theta-1}, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

Derive the UMP test at level $\alpha = 0.05$ for testing $H_0 : \theta \leq 1$ vs $H_1 : \theta > 1$.

(a) Find the critical value for this test explicitly.

(b) Find the power function of this test explicitly.

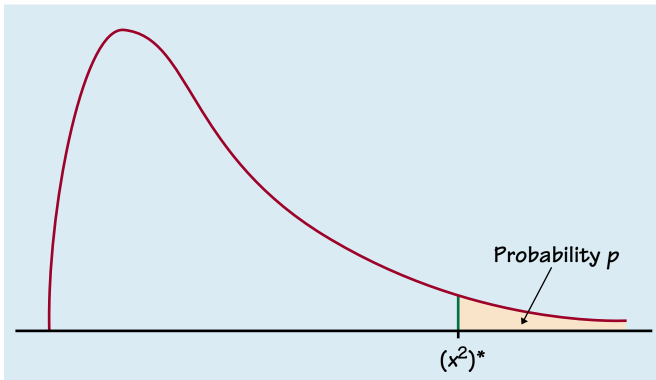

Table entry for p is the point $(X^2)^*$ with probability p lying above it.

TABLE F X^2 distribution critical values

	Tail probability p											
df	.25	.20	.15	.10	.05	.025	.02	.01	.005	.0025	.001	.0005
1	1.32	1.64	2.07	2.71	3.84	5.02	5.41	6.63	7.88	9.14	10.83	12.12
2	2.77	3.22	3.79	4.61	5.99	7.38	7.82	9.21	10.60	11.98	13.82	15.20
3	4.11	4.64	5.32	6.25	7.81	9.35	9.84	11.34	12.84	14.32	16.27	17.73
4	5.39	5.99	6.74	7.78	9.49	11.14	11.67	13.28	14.86	16.42	18.47	20.00
5	6.63	7.29	8.12	9.24	11.07	12.83	13.39	15.09	16.75	18.39	20.51	22.11
6	7.84	8.56	9.45	10.64	12.59	14.45	15.03	16.81	18.55	20.25	22.46	24.10
7	9.04	9.80	10.75	12.02	14.07	16.01	16.62	18.48	20.28	22.04	24.32	26.02
8	10.22	11.03	12.03	13.36	15.51	17.53	18.17	20.09	21.95	23.77	26.12	27.87
9	11.39	12.24	13.29	14.68	16.92	19.02	19.68	21.67	23.59	25.46	27.88	29.67
10	12.55	13.44	14.53	15.99	18.31	20.48	21.16	23.21	25.19	27.11	29.59	31.42
11	13.70	14.63	15.77	17.28	19.68	21.92	22.62	24.72	26.76	28.73	31.26	33.14
12	14.85	15.81	16.99	18.55	21.03	23.34	24.05	26.22	28.30	30.32	32.91	34.82
13	15.98	16.98	18.20	19.81	22.36	24.74	25.47	27.69	29.82	31.88	34.53	36.48
14	17.12	18.15	19.41	21.06	23.68	26.12	26.87	29.14	31.32	33.43	36.12	38.11
15	18.25	19.31	20.60	22.31	25.00	27.49	28.26	30.58	32.80	34.95	37.70	39.72
16	19.37	20.47	21.79	23.54	26.30	28.85	29.63	32.00	34.27	36.46	39.25	41.31
17	20.49	21.61	22.98	24.77	27.59	30.19	31.00	33.41	35.72	37.95	40.79	42.88
18	21.60	22.76	24.16	25.99	28.87	31.53	32.35	34.81	37.16	39.42	42.31	44.43
19	22.72	23.90	25.33	27.20	30.14	32.85	33.69	36.19	38.58	40.88	43.82	45.97
20	23.83	25.04	26.50	28.41	31.41	34.17	35.02	37.57	40.00	42.34	45.31	47.50
21	24.93	26.17	27.66	29.62	32.67	35.48	36.34	38.93	41.40	43.78	46.80	49.01
22	26.04	27.30	28.82	30.81	33.92	36.78	37.66	40.29	42.80	45.20	48.27	50.51
23	27.14	28.43	29.98	32.01	35.17	38.08	38.97	41.64	44.18	46.62	49.73	52.00
24	28.24	29.55	31.13	33.20	36.42	39.36	40.27	42.98	45.56	48.03	51.18	53.48
25	29.34	30.68	32.28	34.38	37.65	40.65	41.57	44.31	46.93	49.44	52.62	54.95
26	30.43	31.79	33.43	35.56	38.89	41.92	42.86	45.64	48.29	50.83	54.05	56.41
27	31.53	32.91	34.57	36.74	40.11	43.19	44.14	46.96	49.64	52.22	55.48	57.86
28	32.62	34.03	35.71	37.92	41.34	44.46	45.42	48.28	50.99	53.59	56.89	59.30
29	33.71	35.14	36.85	39.09	42.56	45.72	46.69	49.59	52.34	54.97	58.30	60.73
30	34.80	36.25	37.99	40.26	43.77	46.98	47.96	50.89	53.67	56.33	59.70	62.16
40	45.62	47.27	49.24	51.81	55.76	59.34	60.44	63.69	66.77	69.70	73.40	76.09
50	56.33	58.16	60.35	63.17	67.50	71.42	72.61	76.15	79.49	82.66	86.66	89.56
60	66.98	68.97	71.34	74.40	79.08	83.30	84.58	88.38	91.95	95.34	99.61	102.7
80	88.13	90.41	93.11	96.58	101.9	106.6	108.1	112.3	116.3	120.1	124.8	128.3
100	109.1	111.7	114.7	118.5	124.3	129.6	131.1	135.8	140.2	144.3	149.4	153.2